GPAP Summer School (2021)

#2 Magnetohydrodynamics (MHD) (M. Brown)

Magnetic forces, induction eq., flux freezing

MHD Equations

- Treat plasma as a single thermodynamic, conducting fluid

- $\frac{F}{\text{vol}} = \frac{ma}{\text{vol}} \to \rho \frac{Dv}{Dt} = J \times B - \nabla p$, where $J = \frac{I}{\text{Area}}$

Steady state $\boxed{J \times B = \nabla p}$

Rivulets of current attract

$V(\underline{r}, t)$ ... eg $P(\underline{r}, t)$ ... $\frac{DP}{Dt} = \left(\frac{\partial P}{\partial t} + v \cdot \nabla P\right)$

$\underline{V}(x, t)$ ... $\frac{dv}{dt} = \frac{dv}{dx}\underbrace{\frac{dx}{dt}}_{v_x} = (v \cdot \nabla v)_x$ Convective derivative

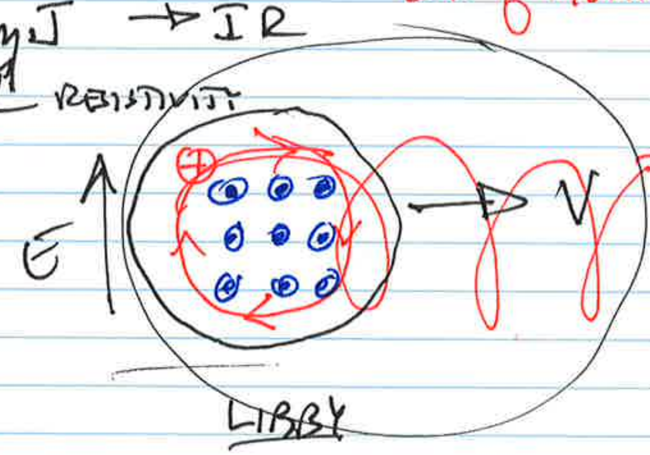
Reason for turbulence ... is $v \cdot \nabla v$ $\boxed{\rho\left(\frac{\partial v}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = \underline{J} \times \underline{B} - \nabla p} + \nu \nabla^2 v$

Eq. of motion

Ohm's Law.

$\oint E \cdot dl = \emptyset$

$E' = \underbrace{E + v \times B}_{\text{Relativity}} = \underbrace{\eta}_{\text{resistivity}} J \to IR$

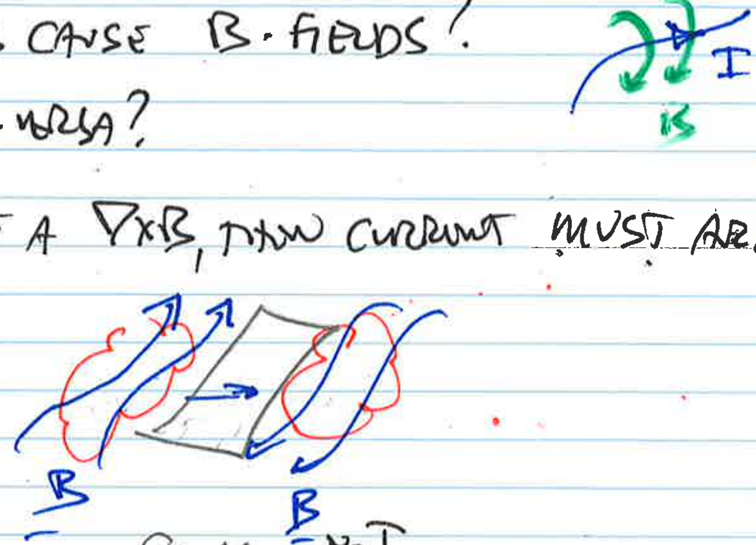


OTHER RELATIONS WE NEED — MKS

MAXWELL'S EQ. AMPERE'S LAW $\boxed{\nabla \times B = \mu_0 J}$

DO CURRENTS CAUSE B-FIELDS? OR VICE-VERSA?

BOTH! IF I CAN MAKE A $\nabla \times B$, THEN CURRENT MUST ARISE!



$$\oint B \cdot dl = \mu_0 I$$

$\boxed{\nabla \cdot B = 0}$ NO MAG. MONOPOLES.

WHY ARE there LARGE SCALE B FIELDS IN the UNIVERSE but no LARGE SCALE E FIELDS?

BECAUSE $\boxed{\nabla \cdot E = \rho/\epsilon_0}$

ASIDE: AMPERE'S LAW

$$\nabla \times B = \mu_0 J + \underbrace{\mu_0 \epsilon_0 \frac{\partial E}{\partial t}}_{\text{DISPLACEMENT CURRENT}}$$

$$= \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

MUCH BIGGER THAN THIS

$$\frac{1}{c^2}\frac{\partial E}{\partial t} \sim \frac{1}{c^2}\frac{vB}{\Delta l / v} = \frac{v^2}{c^2}\frac{B}{\Delta l}$$

$$\nabla \times B \sim \frac{B}{\Delta l}$$

$\frac{1}{c^2}\frac{\partial E}{\partial t}$ is $\frac{v^2}{c^2}$ smaller $\frac{100\,\frac{km}{s}}{3\times10^5\,\frac{km}{s}} = \frac{1}{3000}$

$\sim 10^{-7}$ IN SSX

FARADAY'S LAW $\nabla \times E = -\frac{\partial B}{\partial t}$

$\sim T^{-3/2}$ SMALL

CONSIDER: NOTE OHM'S LAW $E + v \times B = \eta J \cong 0$

$E + v \times B \cong 0$

IDEAL MHD

$\frac{F}{\text{CHARGE}}$ e-FLUID $= (E + v\times B) = \eta J - \frac{1}{n_e} J \times B + \frac{1}{n_e}\nabla P \ldots$

TRY: $\nabla \times (E + v \times B) = \nabla \times (\eta J)$ CURL of OHM'S LAW

$$-\frac{\partial B}{\partial t} + \nabla \times (v \times B) = \eta \nabla \times \left(\frac{\nabla \times B}{\mu_0}\right)$$

related to $v \cdot \nabla B$

RESISTIVITY

$E = \eta J \rightarrow V = RI$

$$\boxed{\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \frac{\eta}{\mu_0}\nabla^2 B}$$

CONVECTION, DISSIPATION or DIFFUSION

IND. EQ. LINEAR IN B

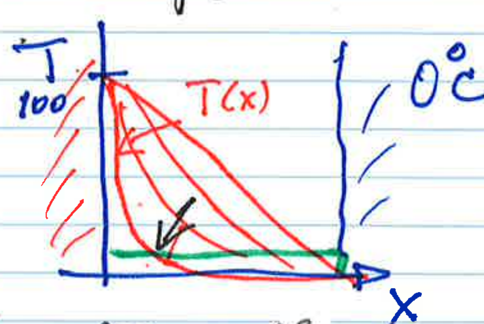
NO VELOCITY ... DIFFUSION EQ. $\frac{\partial B}{\partial t} = \frac{\eta}{\mu_0}\nabla^2 B$

MAG. REYNOLD'S $R_M \equiv \frac{\nabla v B}{\frac{\eta}{\mu_0}\nabla^2 B} = \frac{\text{CONV}}{\text{DIFF}}$

$\nabla \sim \frac{1}{l}$

$$= \frac{\mu_0 l v}{\eta} = \boxed{\mu_0 l v \sigma}$$

eg:



$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$

[illegible]

ASIDE: $\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla p + J \times B + \nu \nabla^2 v$

EQ. of MOTION

FLUID REYNOLD'S $R_e = \frac{\rho v \nabla v}{\nu \nabla^2 v} = \frac{\rho l v}{\nu}$

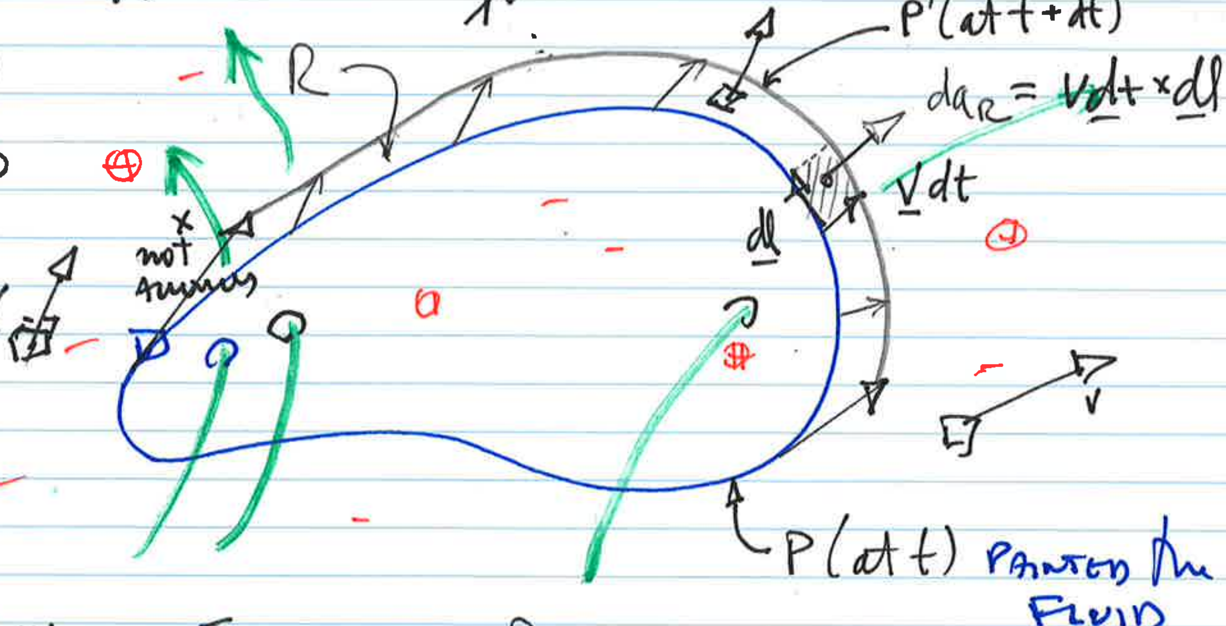
FROZEN-IN FLUX : IF $\eta = 0$, Φ_{MAG} & FLUID MOVE TOGETHER

NEED $E' = E + v \times B = \eta J \sim 0$ (IDEAL) OHM'S LAW

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \cancelto{0}{\frac{\eta}{\mu_0} \nabla^2 B}$$ INDUCTION EQ.

$\nabla \cdot B = 0$

$\oint B \cdot da = 0$ CLOSED VOL.



QUICK WAY: JUST CONSIDER R... RIBBON

$$\Phi_{mag} = \int \underline{B} \cdot \underline{da_r} \quad \text{IN RIBBON}$$

so $$\frac{d\Phi}{dt} = \int \frac{\partial B}{\partial t} \cdot da_r + \int B \cdot \frac{d\,\Delta a_r}{\partial t}$$ but $da_R = v\,dt \times dl$

CHAIN RULE

OHM'S LAW

$$= \int \nabla \times (v \times B) \cdot da_r + \int B \cdot \frac{d\,\Delta a_r}{dt}$$ $\frac{d\,\Delta a_r}{dt} = v \times dl$

recall STOKES $\oint \nabla \times A \cdot da = \oint A \cdot dl$

$$= \oint (v \times B) \cdot dl + \oint^{\text{CLOSED}} B \cdot (v \times dl) = 0$$

BAC-CAB

NO FLUX THROUGH RIBBON.