

# Kunz Lecture Notes for GPAP School on Plasma Physics for Astrophysicists

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## PART I

## Introduction to astrophysical plasmas

## I.1. What is a plasma?

Astrophysical plasmas are remarkably varied, and so it may appear difficult at first to provide a definition of just what constitutes a “plasma”. Is it an ionized, conducting gas? Well, the cold, molecular phase of the interstellar medium has a degree of ionization of  $\lesssim 10^{-6}$ , and yet is considered a plasma. (Indeed, plenty of researchers still model this phase using ideal magnetohydrodynamics!) Okay, so perhaps a sufficiently ionized, conducting gas (setting aside for now what is meant precisely by “sufficiently”)? Well, plasmas don’t necessarily have to be good conductors. Indeed, many frontier topics in plasma astrophysics involve situations in which resistivity is fundamentally important.

Clearly, any definition of a plasma must be accompanied by qualifiers, and these qualifiers are often cast in terms of dimensionless parameters that compare length and time scales. Perhaps the most important dimensionless parameter in the definition of a plasma is the *plasma parameter*,

$$\Lambda \doteq n_e \lambda_D^3, \quad (\text{I.1})$$

where  $n_e$  is the electron number density and

$$\lambda_D \doteq \left( \frac{T}{4\pi e^2 n_e} \right)^{1/2} = 7.4 \left( \frac{T_{\text{eV}}}{n_{\text{cm}^{-3}}} \right)^{1/2} \text{ m} \quad (\text{I.2})$$

is the Debye length. We’ll derive this formula for the Debye length and discuss its physics more in § III.1 of these notes, but for now I’ll simply state its meaning: it is the characteristic length scale on which the Coulomb potential of an individual charged particle is exponentially attenuated (“screened”) by the preferential accumulation (exclusion) of oppositely- (like-) charged particles into (from) its vicinity.<sup>1</sup> Thus,  $\Lambda$  reflects the number of electrons in a Debye sphere. Its dependence upon the temperature  $T$  suggests an alternative interpretation of  $\Lambda$ :

$$\Lambda = \frac{T}{4\pi e^2 / \lambda_D} \sim \frac{\text{kinetic energy}}{\text{potential energy}}. \quad (\text{I.3})$$

Indeed, if the plasma is in thermodynamic equilibrium with a heat bath at temperature  $T$ , then the concentration of discrete charges follows the Boltzmann distribution,

$$n_\alpha(\mathbf{r}) = \bar{n}_\alpha \exp\left(-\frac{q_\alpha \phi(\mathbf{r})}{T}\right), \quad (\text{I.4})$$

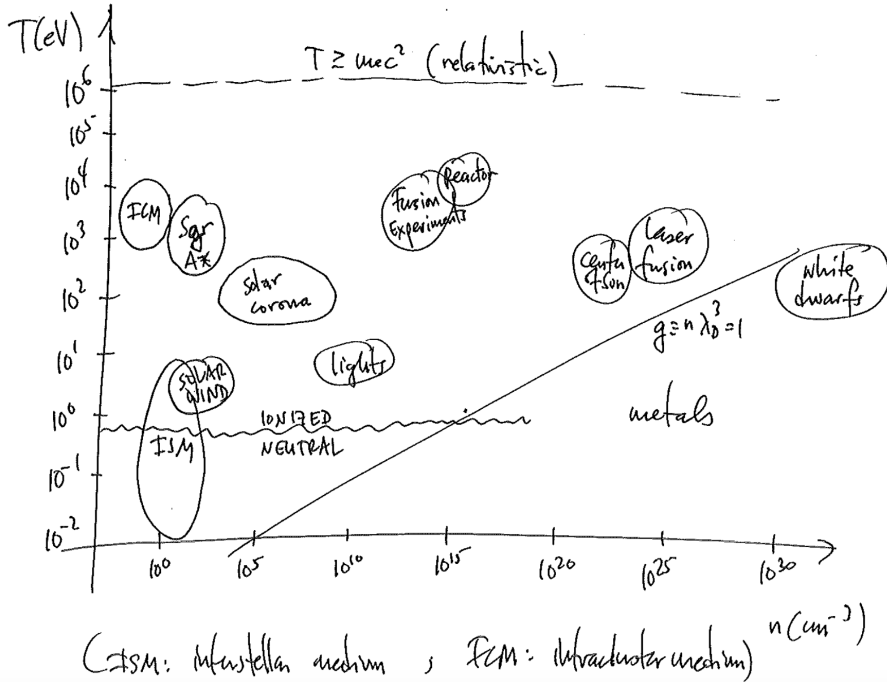
where  $\bar{n}_\alpha$  is the mean number density of species  $\alpha$ ,  $q_\alpha$  is its electric charge, and  $\phi(\mathbf{r})$  is the Coulomb potential. In the limit  $\Lambda \rightarrow \infty$ , the distribution of charges becomes uniform, i.e., the plasma is said to be *quasi-neutral*, with equal amounts of positive and negative charge within a Debye sphere.

Debye shielding is fundamentally due to the polarization of the plasma and the associated redistribution of space charge, and is an example of how a plasma behaves as a dielectric medium. The hotter plasma, the more kinetic energy, the less bound individual electrons are to the protons. When  $\Lambda \gg 1$ , collective electrostatic interactions are much more important than binary particle–particle collisions, and the plasma is said to be

<sup>1</sup>In this course, sometimes temperature will be measured in Kelvin, and sometimes temperature will be measured in energy units (eV) after a hidden multiplication by Boltzmann’s constant  $k_B$ . An energy of 1 eV corresponds to a temperature of  $\sim 10^4$  K (more precisely,  $\simeq 1.16 \times 10^4$  K).

weakly coupled. These are the types of plasmas that we will focus on in this course (e.g., the intracluster medium of galaxy clusters has  $\Lambda \sim 10^{15}$ ).

Shown below is a rogue's gallery of astrophysical and space plasmas in the  $T$ - $n$  plane, with the  $\Lambda = 1$  line indicating a divide between quasi-neutral plasmas (to the left) and metals (to the right):



Clearly, there is a lot of parameter space here and so, to classify these plasmas further, we require additional dimensionless parameters.

## I.2. Fundamental length and time scales

Another useful dividing line between different types of astrophysical and space plasmas is whether they are *collisional* or *collisionless*. In other words, is the mean free path between particle-particle collisions,  $\lambda_{\text{mfp}}$ , larger or smaller than the macroscopic length scales of interest,  $L$ . If  $\lambda_{\text{mfp}} \ll L$ , then the plasma is said to behave as a *fluid*, and various hydrodynamic and magnetohydrodynamic (MHD) equations can be used to describe its evolution. If, on the other hand, the mean free path is comparable to (or perhaps even larger than) the macroscopic length scales of interest, the plasma cannot be considered to be in local thermodynamic equilibrium, and the full six-dimensional phase space (3 spatial coordinates, 3 velocity coordinates) through which the constituent particles move must be retained in the description. Written in terms of the thermal speed of species  $\alpha$ ,

$$v_{\text{th}\alpha} \doteq \left( \frac{2T_\alpha}{m_\alpha} \right)^{1/2}, \quad (\text{I.5})$$

and the collision timescale  $\tau_\alpha$ , the *collisional mean free path* is

$$\lambda_{\text{mfp},\alpha} \doteq v_{\text{th}\alpha} \tau_\alpha. \quad (\text{I.6})$$

For electron–ion collisions,

$$\tau_{ei} = \frac{3\sqrt{m_e}T_e^{3/2}}{4\sqrt{2\pi}n_e\lambda_e Z^2 e^4} \simeq 3.4 \times 10^5 \left( \frac{T_{eV}^{3/2}}{n_{\text{cm}^{-3}}\lambda_e Z^2} \right) \text{ s}, \quad (\text{I.7})$$

where  $Ze$  is the ion charge and  $\lambda_e$  is the electron Coulomb logarithm; for ion–ion collisions,

$$\tau_{ii} = \frac{3\sqrt{m_i}T_i^{3/2}}{4\sqrt{\pi}n_i\lambda_i Z^4 e^4} \simeq 2.1 \times 10^7 \left( \frac{T_{eV}^{3/2}}{n_{\text{cm}^{-3}}\lambda_i Z^4} \right) \text{ s}, \quad (\text{I.8})$$

where  $\lambda_i$  is the ion Coulomb logarithm. Note that the resulting  $\lambda_{\text{mfp},e}$  and  $\lambda_{\text{mfp},i}$  differ only by a factor of order unity:

$$\lambda_{\text{mfp},e} = \frac{3}{4\sqrt{\pi}} \frac{T_e^2}{n_e\lambda_e Z^2 e^4}, \quad \lambda_{\text{mfp},i} = \frac{3\sqrt{2}}{4\sqrt{\pi}} \frac{T_i^2}{n_i\lambda_i Z^4 e^4},$$

and so one often drops the species subscript on  $\lambda_{\text{mfp}}$ . With these definitions, it becomes clear that the plasma parameter (I.1) also reflects the ratio of the mean free path to the Debye length:

$$A \doteq \frac{n_e\lambda_D^4}{\lambda_D} \sim \frac{T_e^2/n_e/e^4}{\lambda_D} \sim \frac{\lambda_{\text{mfp}}}{\lambda_D}; \quad (\text{I.9})$$

again, a measure of the relative importance of collective effects ( $\lambda_D$ ) and binary collisions ( $\lambda_{\text{mfp}}$ ).

Independent of whether a given astrophysical plasma is collisional or collisionless, nearly all such plasmas host magnetic fields, either inherited from the cosmic background in which they reside or produced *in situ* by a dynamo mechanism. There are two ways in which the strength of the magnetic field is quantified. First, the *plasma beta parameter*:

$$\beta_\alpha \doteq \frac{8\pi n_\alpha T_\alpha}{B^2}, \quad (\text{I.10})$$

which reflects the relative energy densities of the thermal motions of the plasma particles and of the magnetic field. Note that

$$\beta_\alpha = \frac{2T_\alpha}{m_\alpha} \times \frac{4\pi m_\alpha n_\alpha}{B^2} = \frac{v_{\text{th}\alpha}^2}{v_{A\alpha}^2}, \quad (\text{I.11})$$

where

$$v_{A\alpha} \doteq \frac{B}{\sqrt{4\pi m_\alpha n_\alpha}} \quad (\text{I.12})$$

is the *Alfvén speed* for species  $\alpha$ .<sup>2</sup> Second, the *plasma magnetization*,  $\rho_\alpha/L$ , where

$$\rho_\alpha \doteq \frac{v_{\text{th}\alpha}}{\Omega_\alpha} \quad (\text{I.13})$$

is the Larmor radius of species  $\alpha$  and

$$\Omega_\alpha \equiv \frac{q_\alpha B}{m_\alpha c} \quad (\text{I.14})$$

is the gyro- (or cyclotron, or Larmor) frequency. What distinguishes many astrophysical plasmas from their terrestrial laboratory counterparts is that the former can have  $\beta \gg 1$  even though  $\rho/L \lll 1$ .<sup>3</sup> In other words, a magnetized astrophysical plasma need not have

<sup>2</sup>Usually, a single Alfvén speed,  $v_A \doteq B/\sqrt{4\pi\rho}$ , is given for a plasma with mass density  $\rho$ .

<sup>3</sup>The  $\sim 5$  keV intracluster medium of galaxy clusters can be magnetized by a magnetic field as weak as  $\sim 10^{-18}$  G.



an energetically important magnetic field, and  $\beta \gg 1$  does not preclude the magnetic field from having dynamical consequences. You've been warned.

There are two more kinetic scales worth mentioning at this point, which we will come to later in this course: the *plasma frequency*,

$$\omega_{p\alpha} = \left( \frac{4\pi n_\alpha e^2}{m_\alpha} \right)^{1/2}, \quad (\text{I.15})$$

and the *skin depth* (or *inertial length*),

$$d_\alpha \doteq \frac{c}{\omega_{p\alpha}} = \left( \frac{m_\alpha c^2}{4\pi n_\alpha e^2} \right)^{1/2}. \quad (\text{I.16})$$

The former is the characteristic frequency at which a plasma oscillates when one sign of charge carriers is displaced from the other sign by a small amount (see §III.2). Indeed, the factor  $(4\pi n_\alpha e^2)$  should look familiar from the definition of the Debye length (see (I.2)). The latter is the characteristic scale below which the inertia of species  $\alpha$  precludes the propagation of (certain) electromagnetic waves. For example, the ion skin depth is the scale at which the ions decouple from the electrons and any fluctuations in which the electrons are taking part (e.g., whistler waves). The following relationship between the skin depth and the Larmor radius may one day come in handy:

$$d_\alpha = \frac{v_{A,\alpha}}{\Omega_\alpha} = \frac{\rho_\alpha}{\beta_\alpha^{1/2}}. \quad (\text{I.17})$$

### I.3. Examples of astrophysical and space plasmas

This part is given as a keynote presentation. Here I simply provide a chart of useful numbers on the next page (ICM = intracluster medium; JET = Joint European Torus, a nuclear fusion experiment; ISM = interstellar medium). For quick reference, the Earth has a  $\sim 0.5$  G magnetic field,  $1 \text{ eV} \sim 10^4 \text{ K}$ ,  $1 \text{ au} \approx 1.5 \times 10^{13} \text{ cm}$ ,  $1 \text{ pc} \approx 3 \times 10^{18} \text{ cm}$ ,  $1 \text{ pc Myr}^{-1} \approx 1 \text{ km s}^{-1}$ .

	Solar wind @ 1 au (Earth location)	ICM @ $\sim 100 \text{ kpc}$ ("cooling radius")	Galactic center @ $0.1 \text{ pc}$ ("Bondi radius")	JET device ( $\sim$ meters)	ISM ("warm")
T	10 eV	$8 \times 10^3 \text{ eV}$	$2 \times 10^3 \text{ eV}$	$10^4 \text{ eV}$	1 eV
n	$10 \text{ cm}^{-3}$	$5 \times 10^{-3} \text{ cm}^{-3}$	$100 \text{ cm}^{-3}$	$10^{14} \text{ cm}^{-3}$	$1 \text{ cm}^{-3}$
B	$100 \mu\text{G}$	$1 \mu\text{G}$	$10^3 \mu\text{G}$	$3 \times 10^4 \text{ G}$	$5 \mu\text{G}$

$V_{Ai}$	40 km/s	1000 km/s	600 km/s	600 km/s	10 km/s
$V_{Ki}$	70 km/s	30 km/s	200 km/s	4000 km/s	10 km/s
$\beta_i \equiv \frac{V_{Ki}^2}{V_{Ai}^2}$	$\sim 0.3-1$	$\sim 10^3$	$\sim 10$	$\sim 0.02$	$\sim 1$
$L$	$\lesssim 1 \text{ au}$	$\sim 10 \text{ kpc} - 100 \text{ kpc}$	$\lesssim 0.1 \text{ pc}$	$\sim 1 \text{ m}$	$\sim 1 \text{ pc} - 100 \text{ pc}$
$\lambda_{\text{ufp}}$	$\sim 0.1 - 1 \text{ au}$	$\sim 0.1 - 10 \text{ kpc}$	$\sim 0.01 \text{ pc}$	$\sim 10 \text{ km}$	$\sim 10^{-7} \text{ pc}$
$\rho_i$	$\sim 10^{-7} \text{ au}$	$\sim 1 \text{ u pc}$	$\sim 1 \text{ p pc}$	$\sim 0.2 \text{ cm}$	$\sim 10^{-11} \text{ pc}$
$\Omega_i$	$\sim 1 \text{ Hz}$	$\sim 0.01 \text{ Hz}$	$\sim 10 \text{ Hz}$	$\sim 300 \text{ MHz}$	$\sim 0.05 \text{ Hz}$

## PART II

## Fundamental of fluid dynamics

Unfortunately, fluid dynamics has all but disappeared from the US undergraduate curriculum, as physics departments have made way for quantum mechanics and condensed matter.<sup>4</sup> This is a shame – yes, it’s classical physics and thus draws less ‘oohs’ and ‘aahs’ from the student (and professorial, for that matter) crowd. But there are many good reasons to study it. First, it forms the bedrock of fascinating and modern topics like non-equilibrium statistical mechanics, including the kinetic theory of gases and particles. Second, it is mathematically rich without being physically opaque. The more you really understand the mathematics, the more you really understand physically what is going on; the same cannot be said for many branches of modern physics. Third, nonlinear dynamics and chaos, burgeoning fields in their own right, are central to arguably the most important unsolved problem in classical physics: fluid turbulence. Solve that, and your solution would have immediate impact and practical benefits to society. Finally, follow in the footsteps of greatness: on Feynman’s chalkboard at the time of his death was the remit ‘to learn . . . nonlinear classical hydro’. With that, let’s begin.

<sup>4</sup>An excellent textbook from which to learn elementary fluid dynamics is Acheson’s *Elementary Fluid Dynamics*. It provides an engaging mix of history, physical insight, and transparent mathematics. I recommend it.

## II.1. The equations of ideal hydrodynamics

The equations of hydrodynamics and MHD may be obtained rigorously by taking velocity-space moments of the Boltzmann and Vlasov–Landau kinetic equations. *Huh? What?* Okay, we’ll get to that soon enough. For now, let’s begin with things that you already know: mass is conserved, Newton’s second law (force equals mass times acceleration), and the first law of thermodynamics (energy is conserved).

### II.1.1. Mass is conserved: The continuity equation

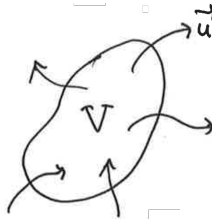
We describe our gaseous fluid by a mass density  $\rho$ , which in general is a function of time  $t$  and position  $\mathbf{r}$ .<sup>5</sup> Imagine an arbitrary volume  $\mathcal{V}$  enclosing some of that fluid. The mass inside of the volume is simply

$$M = \int_{\mathcal{V}} dV \rho. \quad (\text{II.1})$$

Now let’s mathematize our intuition: within this fixed volume, the only way the enclosed mass can change is by material flowing in or out of its surface  $\mathcal{S}$ :

$$\frac{dM}{dt} \doteq \int_{\mathcal{V}} dV \frac{\partial \rho}{\partial t} = - \int_{\mathcal{S}} d\mathbf{S} \cdot \rho \mathbf{u}, \quad (\text{II.2})$$

where  $\mathbf{u}$  is the flow velocity.



Gauss’ theorem may be applied to rewrite the right-hand side of this equation as follows:

$$\int_{\mathcal{S}} d\mathbf{S} \cdot \rho \mathbf{u} = \int_{\mathcal{V}} dV \nabla \cdot (\rho \mathbf{u}). \quad (\text{II.3})$$

Because the volume under consideration is arbitrary, the integrands of the volume integrals in (II.2) and (II.3) must be the same. Therefore,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0} \quad (\text{II.4})$$

This is the *continuity equation*; it’s the differential form of mass conservation.

*Exercise.* Go to the bathroom and turn on the sink slowly to get a nice, laminar stream flowing down from the faucet. Go on, I’ll wait. If you followed instructions, then you’ll see that the stream becomes more narrow as it descends. Knowing that the density of water is very nearly constant, use the continuity equation to show that the cross-sectional area of the stream  $A(z)$  as a function of distance from the faucet  $z$  is

$$A(z) = \frac{A_0}{\sqrt{1 + 2gz/v_0^2}},$$

<sup>5</sup>I sometimes denote the mass density by  $\varrho$  to avoid confusion with the Larmor radius  $\rho$ . But, given that  $\rho$  is standard notation in hydrodynamics for the mass density, and  $\rho$  is standard notation in plasma physics for the Larmor radius, you should learn to tell the difference based on the context.

where  $A_0$  is the cross-sectional area of the stream upon exiting the faucet with velocity  $v_0$  and  $g$  is the gravitational acceleration. If you turn the faucet to make the water flow faster, what happens to the tapering of the stream?

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### II.1.2. Newton's second law: The momentum equation

So far we have an equation for the evolution of the mass density  $\rho$  expressed in terms of the fluid velocity  $\mathbf{u}$ . How does the latter evolve? Newton's second law provides the answer: simply add up the accelerations, divide by the mass (density), and you've got the time rate of change of the velocity. But there is a subtlety here: there is a difference between the time rate of change of the velocity in the lab frame and the time rate of change of the velocity in the fluid frame. So which time derivative of  $\mathbf{u}$  do we take? The key is in how the accelerations are expressed. Are these accelerations acting on a fixed point in space, or are they acting on an element of our fluid? It is much easier (and more physical) to think of these accelerations in the latter sense: given a deformable patch of the fluid – large enough in extent to contain a very large number of atoms but small enough that all the macroscopic variables such as density, velocity, and pressure have a unique value over the dimensions of the patch – what forces are acting on that patch? These are relatively simple to catalog, and we will do so in short order. But first, let's answer our original question: which time derivative of  $\mathbf{u}$  do we take? Since we have committed to expressing the forces in the frame of the fluid element, the acceleration must likewise be expressed in this frame. The acceleration is *not*

$$\frac{\partial \mathbf{u}}{\partial t}. \quad (\text{II.5})$$

Remember what a partial derivative means: something is being fixed! Here, it is the instantaneous position  $\mathbf{r}$  of the fluid element. Equation (II.5) is the answer to the question, 'how does the fluid velocity evolve at a fixed point in space?' Instead, we wish to fix our sights on the fluid element itself, which is moving. The acceleration we calculate must account for this frame transformation:

$$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \nabla \mathbf{u}, \quad (\text{II.6})$$

where  $d\mathbf{r}/dt$  is the rate of change of the position of the fluid element, i.e., the velocity  $\mathbf{u}(t, \mathbf{r})$ . This combination of derivatives is so important that it has its own notation:

$$\frac{D}{Dt} \doteq \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla. \quad (\text{II.7})$$

It is variously referred to as the *Lagrangian derivative*, or comoving derivative, or convective derivative. By contrast, the expression given by (II.5) is the *Eulerian derivative*. Note that the continuity equation (II.4) may be expressed using the Lagrangian derivative as

$$\frac{D \ln \rho}{Dt} = -\nabla \cdot \mathbf{u}, \quad (\text{II.8})$$

which states that incompressible flow corresponds to  $\nabla \cdot \mathbf{u} = 0$ .

So, given some force  $\mathbf{F}$  per unit volume that is acting on our fluid element, we now know how the fluid velocity evolves: force (per unit volume) equals mass (per unit volume) times acceleration (in the frame of the fluid element):

$$\mathbf{F} = \rho \frac{D\mathbf{u}}{Dt}. \quad (\text{II.9})$$

Now we need only catalog the relevant forces. This could be, say, gravity:  $\rho \mathbf{g} = -\rho \nabla \Phi$ .

Or, if the fluid element is conducting, electromagnetic forces (which we'll get to later in the course). But the most deserving of discussion at this point is the pressure force due to the internal thermal motions of the particles comprising the gas. For an ideal gas, the equation of state is

$$P = \frac{\rho k_B T}{m} \doteq \rho C^2, \quad (\text{II.10})$$

where  $T$  is the temperature in Kelvin,  $k_B$  is the Boltzmann constant,  $m$  is the mass per particle, and  $C$  is the speed of sound in an isothermal gas. Plasma physicists often drop Boltzmann's constant and register temperature in energy units (e.g., eV), and I will henceforth do the same in these notes. How does gas pressure due to microscopic particle motions exert a macroscopic force on a fluid element? First, the pressure must be spatially non-uniform: there must be more or less energetic content in the thermal motions of the particles in one region versus another, whether it be because the gas temperature varies in space or because there are more particles in one location as opposed to another. For example, the pressure force in the  $x$  direction in a slab of thickness  $dx$  and cross-sectional area  $dy dz$  is

$$[P(t, x - dx/2, y, z) - P(t, x + dx/2, y, z)] dy dz = -\frac{\partial P}{\partial x} dV. \quad (\text{II.11})$$

Unless the thermal motions of the particles are not sufficiently randomized to be isotropic (e.g., if the collisional mean free path of the plasma is so long that inter-particle collisions cannot drive the system quickly enough towards local thermodynamic equilibrium), there is nothing particularly special about the  $x$  direction, and so the pressure force acting on some differential volume  $dV$  is just  $-\nabla P dV$ .

Assembling the lessons we've learned here, we have the following force equation for our fluid:

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} \doteq \rho \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = -\nabla P - \rho \nabla \Phi} \quad (\text{II.12})$$

This equation is colloquially known as the *momentum equation*, even though it evolves the fluid velocity rather than its momentum density. To obtain an equation for the latter, the continuity equation (II.4) may be used to move the mass density into the time and space derivatives:

$$\begin{aligned} \frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla \cdot (\rho\mathbf{u}\mathbf{u}) &= \frac{\partial\rho}{\partial t}\mathbf{u} + \rho \frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u} \cdot \nabla\mathbf{u} + \nabla \cdot (\rho\mathbf{u})\mathbf{u} \\ &= \left[ \frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) \right] \mathbf{u} + \rho \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} \\ &= \left[ \begin{array}{c} 0 \end{array} \right] \mathbf{u} + \rho \frac{D\mathbf{u}}{Dt} = \mathbf{F}. \end{aligned} \quad (\text{II.13})$$

Thus, an equation for the momentum density:

$$\boxed{\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla \cdot (\rho\mathbf{u}\mathbf{u}) = -\nabla P - \rho \nabla \Phi} \quad (\text{II.14})$$

This form is particularly useful for deriving an evolution equation for the kinetic energy density. Dotted (II.14) with  $\mathbf{u}$  and grouping terms,

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 \right) + \nabla \cdot \left( \frac{1}{2} \rho u^2 \mathbf{u} \right) = -\mathbf{u} \cdot \nabla P - \rho \mathbf{u} \cdot \nabla \Phi, \quad (\text{II.15})$$

which is a statement that the kinetic energy of a fluid element changes as work is done by the forces.

Now, how do we know the pressure  $P$ ? There's an equation for that...

### II.1.3. First law of thermodynamics: The internal energy equation

There are several ways to go about obtaining an evolution equation for the pressure. One way is to introduce the *internal energy*,

$$e \doteq \frac{P}{\gamma - 1} \quad (\text{II.16})$$

and use the first law of thermodynamics to argue that  $e$  is conserved but for  $P dV$  work:

$$\boxed{\frac{\partial e}{\partial t} + \nabla \cdot (e\mathbf{u}) = -P\nabla \cdot \mathbf{u}} \quad (\text{II.17})$$

This is the *internal energy* equation.

Equation (II.17) may be used to derive a total (kinetic + internal + potential) energy equation for the fluid as follows. Do (II.15) + (II.17):

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 + e \right) + \nabla \cdot \left[ \left( \frac{1}{2} \rho u^2 + e \right) \mathbf{u} \right] &= -\nabla \cdot (P\mathbf{u}) - \rho \mathbf{u} \cdot \nabla \Phi, \\ &= -(\gamma - 1) \nabla \cdot (e\mathbf{u}) - \rho \mathbf{u} \cdot \nabla \Phi \\ \implies \frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 + e \right) + \nabla \cdot \left[ \left( \frac{1}{2} \rho u^2 + \gamma e \right) \mathbf{u} \right] &= -\rho \mathbf{u} \cdot \nabla \Phi. \end{aligned} \quad (\text{II.18})$$

Now use the continuity equation (II.4) to write

$$\frac{\partial(\rho\Phi)}{\partial t} + \nabla \cdot (\rho\Phi\mathbf{u}) = \rho \mathbf{u} \cdot \nabla \Phi + \rho \frac{\partial \Phi}{\partial t}. \quad (\text{II.19})$$

Adding this equation to (II.18) yields the desired result:

$$\boxed{\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 + e + \rho\Phi \right) + \nabla \cdot \left[ \left( \frac{1}{2} \rho u^2 + \gamma e + \rho\Phi \right) \mathbf{u} \right]} = \rho \frac{\partial \Phi}{\partial t} \quad (\text{II.20})$$

The first term in parentheses under the time derivative is sometimes denoted by  $\mathcal{E}$ .

Yet another way of expressing the internal energy equation (II.17) is to write  $e = \rho T/m(\gamma - 1)$  and use the continuity equation (II.4) to eliminate the derivatives of the mass density. The result is

$$\frac{D \ln T}{Dt} = -(\gamma - 1) \nabla \cdot \mathbf{u}, \quad (\text{II.21})$$

which states that the temperature of a fluid element is constant in an incompressible fluid (*viz.*, one with  $\nabla \cdot \mathbf{u} = 0$ ). If this seems intuitively unfamiliar to you, consider this: the hydrodynamic entropy of a fluid element is given by

$$s \doteq \frac{1}{\gamma - 1} \ln P \rho^{-\gamma} = \frac{1}{\gamma - 1} \ln T \rho^{1-\gamma}. \quad (\text{II.22})$$

Taking the Lagrangian time derivative of the entropy along the path of a fluid element yields

$$\frac{Ds}{Dt} = \frac{D \ln T}{Dt} - (\gamma - 1) \frac{D \ln \rho}{Dt}. \quad (\text{II.23})$$

It is then just a short trip back to (II.8) to see that (II.21) is, in fact, the second law of thermodynamics – entropy is conserved in the absence of sources or dissipative sinks:

$$\boxed{\frac{Ds}{Dt} = 0} \quad (\text{II.24})$$

## II.2. Summary: Adiabatic equations of hydrodynamics

The adiabatic equations of hydrodynamics, written in conservative form, are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (\text{II.25a})$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla P - \rho \nabla \Phi, \quad (\text{II.25b})$$

$$\frac{\partial e}{\partial t} + \nabla \cdot (e \mathbf{u}) = -P \nabla \cdot \mathbf{u}. \quad (\text{II.25c})$$

The left-hand sides of these equations express advection of, respectively, the mass density, the momentum density, and the internal energy density by the fluid velocity; the right-hand sides represents sources and sinks. If we instead write these equations in terms of the density, fluid velocity, and entropy and make use of the Lagrangian derivative (II.7), we have

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}, \quad (\text{II.25d})$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla P - \nabla \Phi, \quad (\text{II.25e})$$

$$\frac{Ds}{Dt} = 0, \quad (\text{II.25f})$$

where  $s \doteq (\gamma - 1)^{-1} \ln P \rho^{-\gamma}$ . The limit  $\gamma \rightarrow \infty$ , often of utility for describing liquids, corresponds to  $D\rho/Dt = 0$ , i.e., incompressibility.

## II.3. Mathematical matters

The nonlinear combination  $\mathbf{u} \cdot \nabla \mathbf{u}$  that features prominently in the Lagrangian time derivative can be complicated, particularly in curvilinear coordinates where the gradient operator within it acts on the unit vectors within  $\mathbf{u}$ . For example, in cylindrical coordinates  $(R, \varphi, Z)$ ,

$$\begin{aligned} \mathbf{u} \cdot \nabla \mathbf{u} &= \mathbf{u} \cdot \nabla (u_R \hat{\mathbf{R}} + u_\varphi \hat{\boldsymbol{\varphi}} + u_Z \hat{\mathbf{Z}}) \\ &= (\mathbf{u} \cdot \nabla u_R) \hat{\mathbf{R}} + (\mathbf{u} \cdot \nabla u_\varphi) \hat{\boldsymbol{\varphi}} + (\mathbf{u} \cdot \nabla u_Z) \hat{\mathbf{Z}} + \frac{u_\varphi^2}{R} \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \varphi} + \frac{u_R u_\varphi}{R} \frac{\partial \hat{\mathbf{R}}}{\partial \varphi} \\ &= (\mathbf{u} \cdot \nabla u_i) \hat{\mathbf{e}}_i - \frac{u_\varphi^2}{R} \hat{\mathbf{R}} + \frac{u_R u_\varphi}{R} \hat{\boldsymbol{\varphi}}, \end{aligned} \quad (\text{II.26})$$

where, to obtain the final equality, we have used  $\partial \hat{\boldsymbol{\varphi}} / \partial \varphi = -\hat{\mathbf{R}}$  and  $\partial \hat{\mathbf{R}} / \partial \varphi = \hat{\boldsymbol{\varphi}}$ ; summation over the repeated index  $i$  is implied in the first term in the final line. A

similar calculation in spherical coordinates  $(r, \vartheta, \varphi)$  yields

$$\begin{aligned} \mathbf{u} \cdot \nabla \mathbf{u} &= \left( u_r \frac{\partial}{\partial r} + \frac{u_\vartheta}{r} \frac{\partial}{\partial \vartheta} + \frac{u_\varphi}{r \sin \vartheta} \frac{\partial}{\partial \varphi} \right) (u_r \hat{\mathbf{r}} + u_\vartheta \hat{\boldsymbol{\vartheta}} + u_\varphi \hat{\boldsymbol{\varphi}}) \\ &= (\mathbf{u} \cdot \nabla u_i) \hat{\mathbf{e}}_i - \frac{u_\vartheta^2 + u_\varphi^2}{r} \hat{\mathbf{r}} + \left( \frac{u_r u_\vartheta}{r} - \frac{u_\varphi^2 \cot \vartheta}{r} \right) \hat{\boldsymbol{\vartheta}} + \left( \frac{u_\vartheta u_\varphi \cot \vartheta}{r} + \frac{u_r u_\varphi}{r} \right) \hat{\boldsymbol{\varphi}} \end{aligned} \quad (\text{II.27})$$

The last two terms in the cylindrical  $\mathbf{u} \cdot \nabla \mathbf{u}$ , equation (II.26), might look familiar to you from working in rotating frames. Indeed, let us write  $\mathbf{u} = \mathbf{v} + R\Omega(R, Z)\hat{\boldsymbol{\varphi}}$ , where  $\Omega$  is an angular velocity, and substitute this decomposition into (II.26):

$$\begin{aligned} \mathbf{u} \cdot \nabla \mathbf{u} &= [(\mathbf{v} + R\Omega\hat{\boldsymbol{\varphi}}) \cdot \nabla v_i] \hat{\mathbf{e}}_i + [(\mathbf{v} + R\Omega\hat{\boldsymbol{\varphi}}) \cdot \nabla (R\Omega)] \hat{\boldsymbol{\varphi}} \\ &\quad - \frac{(v_\varphi + R\Omega)^2}{R} \hat{\mathbf{R}} + \frac{v_R(v_\varphi + R\Omega)}{R} \hat{\boldsymbol{\varphi}} \\ &= \left[ \left( \mathbf{v} \cdot \nabla + \Omega \frac{\partial}{\partial \vartheta} \right) v_i \right] \hat{\mathbf{e}}_i + \left[ 2\Omega \hat{\mathbf{z}} \times \mathbf{v} - R\Omega^2 \hat{\mathbf{R}} + R\hat{\boldsymbol{\varphi}}(\mathbf{v} \cdot \nabla)\Omega \right] \\ &\quad + \left[ \frac{v_R v_\varphi}{R} \hat{\boldsymbol{\varphi}} - \frac{v_\varphi^2}{R} \hat{\mathbf{R}} \right]. \end{aligned} \quad (\text{II.28})$$

Each of these terms has a straightforward physical interpretation. The first term in brackets represents advection by the flow and the rotation. The second term in brackets contains the Coriolis force, the centrifugal force, and ‘tidal’ terms due to the differential rotation, in that order. (The ‘tidal’ terms can be thought of the fictitious acceleration required for a fluid element to maintain its presence in the local rotating frame as it is displaced radially or vertically. They come from Taylor expanding the angular velocity about a point in the disk.) The third and final term in brackets are curvature terms due to the cylindrical geometry.

As a close to this section, let me advise you to brush up on your vector calculus. . .

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}), \\ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}), \\ \nabla \times (\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{A}) + \mathbf{A}(\nabla \cdot \mathbf{B}), \\ &\dots \end{aligned}$$

Fluid dynamics is full of these things, and you should either (i) commit them to memory, (ii) carry your NRL formulary with you everywhere, or (iii) know how to quickly derive them using things like

$$\epsilon_{kij} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl},$$

where  $\delta_{ij}$  is the Kronecker delta and  $\epsilon_{ijk}$  is the Levi-Civita symbol.

Exercise: Show that the  $R\varphi$ -component in cylindrical coordinates of the rate-of-strain tensor

$$W_{ij} \doteq \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k}$$

is given by

$$W_{R\varphi} = \frac{1}{R} \frac{\partial u_R}{\partial \varphi} + R \frac{\partial}{\partial R} \frac{u_\varphi}{R}.$$



## PART III

# Fundamentals of plasmas

Now that we have the fluid equations under our belts, let us discuss why we might expect them to apply to a plasma (instead of the more familiar fluid). There are three concepts to cover in this regard: Debye shielding and quasi-neutrality, plasma oscillations, and collisional relaxation of the plasma to take on a Maxwell–Boltzmann distribution of particle velocities.

### III.1. Debye shielding and quasi-neutrality

In § I.1, we mentioned the concept of the *Debye length* and explained its importance in the definition of a plasma. Here we actually derive it from first principles. This derivation starts by recalling that a large plasma parameter  $\Lambda \gg 1$  implies that the kinetic energy of the plasma particles is much greater than the potential energy due to Coulomb interactions amongst binary pairs of particles. In this case, the plasma temperature  $T$  is much bigger than the Coulomb energy  $e\phi \sim e^2/\Delta r \sim e^2 n^{1/3}$ , where  $\phi$  is the electrostatic potential,  $\Delta r \sim n^{-1/3}$  is the typical interparticle distance, and  $n$  is the number density of the particles. Assuming a plasma in local thermodynamic equilibrium, the number density of species  $\alpha'$  with charge  $q_{\alpha'}$  sitting in the potential  $\phi_{\alpha}$  of one ‘central’ particle of species  $\alpha$  ought to satisfy the Boltzmann relation

$$n_{\alpha'}(\mathbf{r}) = \bar{n}_{\alpha'} \exp\left(-\frac{q_{\alpha'}\phi_{\alpha}(\mathbf{r})}{T}\right) \approx \bar{n}_{\alpha'} \left(1 - \frac{q_{\alpha'}\phi_{\alpha}(\mathbf{r})}{T}\right), \quad (\text{III.1})$$

where the potential  $\phi_{\alpha}(\mathbf{r})$  depends on the distance  $\mathbf{r}$  from the ‘central’ particle. To obtain the approximate equality, we have used the assumption  $T \gg e\phi_{\alpha}$  to Taylor expand the Boltzmann factor in its small argument. Inserting (III.1) into the Gauss–Poisson law for the electric field  $\mathbf{E} = -\nabla\phi_{\alpha}$ , we have

$$\begin{aligned} \nabla \cdot \mathbf{E} &= -\nabla^2\phi_{\alpha} = 4\pi q_{\alpha}\delta(\mathbf{r}) + 4\pi \sum_{\alpha'} q_{\alpha'} n_{\alpha'} \\ &\approx 4\pi q_{\alpha}\delta(\mathbf{r}) + 4\pi \sum_{\alpha'} q_{\alpha'} \bar{n}_{\alpha'} - \underbrace{\left(\sum_{\alpha'} \frac{4\pi \bar{n}_{\alpha'} q_{\alpha'}^2}{T}\right)}_{\doteq \lambda_{\text{D}}^{-2}} \phi_{\alpha}. \end{aligned} \quad (\text{III.2})$$

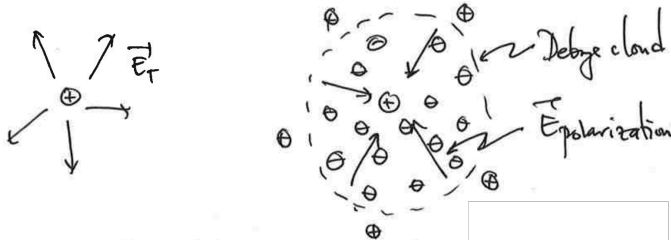
The first term in (III.2) is the point-like charge of the ‘central’ particle located at  $\mathbf{r} = \mathbf{0}$ . The second term is the sum over all charges in the plasma, and equals zero if the plasma is overall charge-neutral (as it should be). The final term introduces the Debye length (see (I.2)), which is the only characteristic scale in (III.2). Note further that this equation has no preferred direction, and so we may exploit its spherical symmetry to recast it as follows:

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial \phi_{\alpha}}{\partial r} - \frac{1}{\lambda_{\text{D}}^2} \phi_{\alpha} = 4\pi q_{\alpha} \delta(\mathbf{r}). \quad (\text{III.3})$$

The solution to this equation that asymptotes to the Coulomb potential  $\phi_{\alpha} \rightarrow q_{\alpha}/r$  as  $r \rightarrow 0$  and to zero as  $r \rightarrow \infty$  is

$$\boxed{\phi_{\alpha} = \frac{q_{\alpha}}{r} \exp\left(-\frac{r}{\lambda_{\text{D}}}\right)} \quad (\text{III.4})$$

This equation states that the bare potential of the ‘central’ charge is exponentially attenuated (‘shielded’) on typical distances  $\sim \lambda_D$ . This is *Debye shielding*, and the sphere of neutralizing charge accompanying the ‘central’ charge is referred to as the *Debye sphere* (or cloud). Debye shielding of an ion by preferential accumulation of electrons in its vicinity is sketched below:



Note that the electric field due to the polarization of the plasma in response to the ion’s bare Coulomb potential acts in the opposite direction to the unshielded electric field.

Now, there was nothing particularly special about the charge that we singled out as our ‘central’ charge. Indeed, we could have performed the above integration for any charge in the plasma. This leads us to the fundamental tenet in the statistical mechanics of a weakly coupled plasma with  $\Lambda \gg 1$ : every charge simultaneously hosts its own Debye sphere while being a member of another charge’s Debye sphere. The key points are that, by involving a huge number of particles in the small-scale electrostatics of the plasma, these Coulomb-mediated relations (i) make the plasma ‘quasi-neutral’ on scales  $\gg \lambda_D$  and (ii) make collective effects in the plasma much more important than individual binary effects due to particle-particle pairings. The latter is what makes a plasma very different from a neutral gas, in which particle-particle interactions occur through hard-body collisions on scales comparable to the mean particle size.

One consequence of Debye shielding is that the electric fields that act on large scales due to the self-consistent collective interactions between  $\sim \Lambda$  Debye clouds are smoothly varying in space and time. As a result, when we write down Maxwell’s equations for our quasi-neutral plasma, the fields that appear are these smooth, coarse-grained fields whose spatial structure resides far above the Debye length. Mathematically, we average the Maxwell equations over the microscopic (i.e., Debye) scales, and what remains are the collective macroscopic fields that ultimately make their way into the magnetohydrodynamics of the plasma ‘fluid’.

### III.2. Plasma oscillations

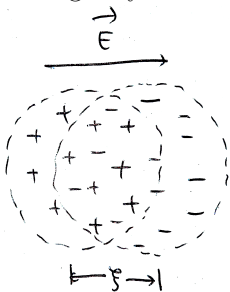
In the previous section, we spoke of a characteristic length scale below which particle-particle interactions are important and above which they are supplanted by collective effects between a large number of quasi-neutral Debye spheres. Is there a corresponding characteristic time scale? The answer is yes, and it may be obtained simply by dimensional analysis: take our Debye length and divide by a velocity to get time. The only velocity in our plasma thus far is the thermal speed,  $v_{th\alpha} = \sqrt{2T/m_\alpha}$ , and so that must be it... we have obtained the *plasma frequency* of species  $\alpha$ ,

$$\omega_{p\alpha} \doteq \sqrt{\frac{4\pi q_\alpha^2 n_\alpha}{m_\alpha}} \sim \frac{\lambda_D}{v_{th\alpha}}. \quad (\text{III.5})$$

Of particular importance, given the smallness of the electron mass, is the electron plasma frequency  $\omega_{pe}$ , which is  $\sim \sqrt{m_i/m_e}$  larger than the ion plasma frequency and is generally the largest frequency in a weakly coupled plasma.

Fine. Dimensional analysis works. But what does this frequency actually mean? Go back to our picture of Debye shielding. That was a static picture, in that we waited long enough for the plasma to settle down into charge distributions governed by Boltzmann relations. What if we didn't wait? Surely there was some transient process whereby the particles moved around to configure themselves into these nice equilibrated Debye clouds. There was, and this transient process is referred to as a *plasma oscillation*, and it has a characteristic frequency of (you guessed it)  $\omega_{pe}$ . Let's show this.

Imagine a spatially uniform, quasi-neutral plasma with well-equilibrated Debye clouds. Shift all of the electrons slightly to the right by a distance  $\xi$ , as shown in the figure below:



The offset between the electrons and the ions will cause an electric field pointing from the ions to the displaced electrons, given by  $E = 4\pi n_e \xi$ . The equation of motion for the electrons is then

$$m_e \frac{d^2 \xi}{dt^2} = -eE = -4\pi e^2 n_e \xi = -m_e \omega_{pe}^2 \xi \quad \implies \quad \frac{d^2 \xi}{dt^2} = -\omega_{pe}^2 \xi. \quad (\text{III.6})$$

This is just the equation for a simple harmonic oscillator with frequency  $\omega_{pe}$ . So, small displacements between oppositely charged species result in *plasma oscillations* (or ‘Langmuir oscillations’), a collective process that occurs as the plasma attempts to restore quasi-neutrality in response to some disturbance. Retaining the effects of electron pressure makes these oscillations propagate dispersively with a non-zero group velocity; these *Langmuir waves* have the dispersion relation  $\omega^2 \approx \omega_{pe}^2 (1 + 3k^2 \lambda_{De}^2)$ , where  $k$  is the wavenumber of the perturbation. More on that later.

### III.3. Collisional relaxation and the Maxwell–Boltzmann distribution

In order for the plasma particles to move freely as plasma oscillations attempt to set up equilibrated Debye clouds, the mean free path between particle–particle collisions must be larger than the Debye length. We may estimate the former in term of the collision cross-section  $\sigma$ ,

$$\lambda_{\text{mfp}} \sim \frac{1}{n\sigma} \sim \frac{T^2}{ne^4},$$

where the cross-section  $\sigma = \pi b^2$  is given by a balance between the Coulomb potential energy,  $\sim e^2/b$ , across some typical impact parameter  $b$  and the kinetic energy of the particles,  $\sim T$ . Comparing this mean free path to the Debye length (I.2), we find

$$\frac{\lambda_{\text{mfp}}}{\lambda_D} \sim \frac{T^2}{ne^4} \left( \frac{ne^2}{T} \right)^{1/2} \sim n\lambda_D^3 \doteq \Lambda \gg 1.$$

Thus, a particle can travel a long distance and experience the macroscopic fields exerted by the collective electrodynamics of the plasma before being deflected by much the

shorter-range, microscopic electric fields generated by another individual particle (recall (I.9)).

The scale separation between the collisional mean free path and the Debye length due to the enormity of the plasma parameter in a weakly coupled plasma says something very important about the statistical mechanics of the plasma. Because  $\lambda_{\text{mfp}}/\lambda_{\text{D}} \sim \omega_{\text{pe}}\tau_{\text{ei}} \gg 1$ , the particle motions are randomized and the velocity distribution of the plasma particles relaxes to a local Maxwell–Boltzmann distribution on (collisional) timescales that are much longer than the timescale on which particle correlations are established and Coulomb potentials are shielded. As a result, collisions in the plasma occur between partially equilibrated Debye clouds instead of between individual particles, the mathematical result being that the ratio  $\lambda_{\text{mfp}}/\lambda_{\text{D}}$  is attenuated by a factor  $\sim \ln \Lambda \approx 10\text{--}40$ . Thus, the logarithmic factors in the collision times (I.7) and (I.8).

Now, about this collisional relaxation. This school isn't the place to go through all the details of how collision operators are derived, but we need to establish a few facts. First, because of Debye shielding, the vast majority of scatterings that a particle experiences as it moves through a plasma are *small-angle scatterings*, with each event changing the trajectory of a particle by a small amount. These accumulate like a random walk in angle away from the original trajectory of the particle, with an average deflection angle  $\langle \theta \rangle = 0$  but with a mean-square deflection angle  $\langle \theta^2 \rangle$  proportional to the number of scattering events. For a typical electron scattering off a sea of Debye-shielded ions of charge  $Ze$  and density  $n$ , this angle satisfies

$$\langle \theta^2 \rangle \approx \frac{8\pi n L Z^2 e^4}{m_e^2 v_{\text{the}}^4} \ln \Lambda \quad (\text{III.7})$$

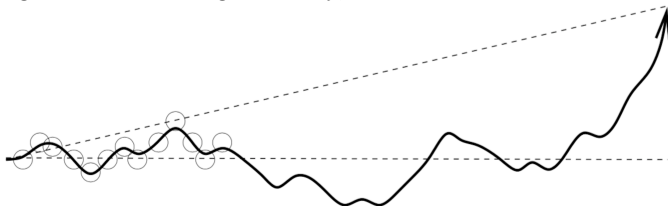
after the electron has traversed a distance  $L$ . A large deflection angle, i.e.  $\langle \theta^2 \rangle \sim 1$ , is reached once this distance

$$L \sim \frac{m_e^2 v_{\text{the}}^4}{8\pi n Z^2 e^4} \frac{1}{\ln \Lambda} \sim v_{\text{the}} \tau_{\text{ei}} \doteq \lambda_{\text{mfp},e}, \quad (\text{III.8})$$

the collisional mean free path (recall the definition of the electron–ion collision time, equation (I.7)). Noting that the impact parameter for a single 90-degree scattering is  $\sim Ze^2/T$ , we find the ratio of the cross-section for many small-angle scatterings to accumulate a 90-degree deflection,  $\sigma_{\text{multi},90^\circ} \sim 1/nL$  using (III.8), to the cross-section for a single 90-degree scattering,  $\sigma_{\text{single},90^\circ} = \pi b^2$  with  $b \sim Ze^2/T$ , is

$$\frac{\sigma_{\text{multi},90^\circ}}{\sigma_{\text{single},90^\circ}} \sim \ln \Lambda \gg 1. \quad (\text{III.9})$$

Thus, in a weakly coupled plasma, multiple small-angle scatterings are more important than a single large-scale scattering. Visually,



This is the physical origin of the  $\ln \Lambda$  reduction in collision time mentioned in the prior paragraph.

So what do these collisions mean for treating our plasma as a fluid? If  $\lambda_{\text{mfp}}$  is much less than any other macroscopic scale of dynamical interest (i.e., scales on which hydrodynamics occurs), then the *velocity distribution function*  $f(\mathbf{v})$  of the plasma – that

is, the differential number of particles with velocities between  $\mathbf{v}$  and  $\mathbf{v} + d\mathbf{v}$  – is well described by a Maxwell–Boltzmann distribution (often simply called a ‘Maxwellian’):

$$f_M(v) \doteq \frac{n}{\pi^{3/2} v_{\text{th}}^3} \exp\left(-\frac{v^2}{v_{\text{th}}^2}\right). \quad (\text{III.10})$$

The factor of  $\pi^{3/2} v_{\text{th}}^3$  is there for normalization purposes:

$$\int d^3\mathbf{v} f_M(\mathbf{v}) = 4\pi \int dv v^2 f_M(v) = n \quad (\text{III.11})$$

is the number of particles per unit volume. (Any particle distribution function should satisfy this constraint.) Note that the Maxwellian is isotropic in velocity space, depending only on the speed of the particles (rather than their vector velocity). If these particles are all co-moving with some bulk velocity  $\mathbf{u}$ , then this ‘fluid’ velocity is subtracted off to ensure an isotropic distribution function in that ‘fluid’ frame:

$$f_M(\mathbf{v}) \doteq \frac{n}{\pi^{3/2} v_{\text{th}}^3} \exp\left(-\frac{|\mathbf{v} - \mathbf{u}|^2}{v_{\text{th}}^2}\right). \quad (\text{III.12})$$

Note that the first moment of this distribution

$$\int d^3\mathbf{v} \mathbf{v} f_M(\mathbf{v}) = n\mathbf{u}; \quad (\text{III.13})$$

and that the (mass-weighted) second moment of this distribution

$$\int d^3\mathbf{v} m |\mathbf{v} - \mathbf{u}|^2 f_M(\mathbf{v}) = 3P. \quad (\text{III.14})$$

(Again, any velocity distribution function should satisfy these constraints.)

Different species collisionally relax to a Maxwellian at different rates (e.g.,  $\tau_{ee} \sim \tau_{ei} \sim \sqrt{m_i/m_e} \tau_{ii} \sim (m_i/m_e) \tau_{ie}$ ), and so each species may be described by their own Maxwellians:

$$f_{M,\alpha}(\mathbf{v}) \doteq \frac{n_\alpha}{\pi^{3/2} v_{\text{th}\alpha}^3} \exp\left(-\frac{|\mathbf{v} - \mathbf{u}_\alpha|^2}{v_{\text{th}\alpha}^2}\right). \quad (\text{III.15})$$

But, in the long-time limit, unless some process actively dis-equilibrates the species on a timescale comparable to or smaller than these collision times, all species will take on the *same*  $\mathbf{u}$  and the *same*  $T$ . Their densities are, of course, the same as well, as guaranteed by quasi-neutrality (*viz.*,  $\omega_{pe}\tau \gg 1$  for all collision times  $\tau$ ).

Note then, that when we wrote down our hydrodynamic equations for a scalar pressure (see ((II.14)) and ((II.17))) and didn’t affix any species labels to any quantities, we were implicitly assuming that our hydrodynamics occurs on time scales much longer than the collisional equilibration times, so that all species can be well described by local Maxwellians with the same density, fluid velocity, and temperature. Not all astrophysical systems are so cooperative, and anisotropic pressures, velocity drifts between species, and dis-equilibration of species temperatures can often be the norm. Yes, hydrodynamics and MHD are fairly simple, but do not let their simplicity lure you into using them when it’s not appropriate to do so – a hard-earned lesson for many astrophysicists.

## PART IV

## Linear theory of MHD instabilities

Prof. Klein did some MHD linear waves. Now let's do some MHD linear instabilities. Start with the ideal-MHD equations:

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}, \quad (\text{IV.1a})$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla \left( P + \frac{B^2}{8\pi} \right) + \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{4\pi\rho} - \nabla\Phi, \quad (\text{IV.1b})$$

$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B} \cdot \nabla) \mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u}), \quad (\text{IV.1c})$$

$$\frac{P}{\gamma - 1} \frac{D}{Dt} \ln \frac{P}{\rho^\gamma} = 0, \quad (\text{IV.1d})$$

where  $D/Dt \doteq \partial/\partial t + \mathbf{u} \cdot \nabla$  is the Lagrangian derivative. A rarely publicized but useful form of the induction equation (IV.1c) is obtained by defining the magnetic-field unit vector  $\hat{\mathbf{b}} \doteq \mathbf{B}/B$  and writing separate equations for it and the magnetic-field strength  $B$ :

$$\frac{D \ln B}{Dt} = (\hat{\mathbf{b}} \hat{\mathbf{b}} - \mathbf{I}) : \nabla \mathbf{u} \quad \text{and} \quad \frac{D \hat{\mathbf{b}}}{Dt} = (\mathbf{I} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : (\hat{\mathbf{b}} \cdot \nabla \mathbf{u}). \quad (\text{IV.2})$$

Just thought I'd throw that out there for you to chew on.

The program is to set up some equilibria and then subject them to small-amplitude perturbations in the fluid and magnetic field. There are a few different ways of doing this and assessing whether the system is stable or unstable to these perturbations. There's something called the MHD energy principle, which will tell you whether a given set of perturbations about some equilibrium state will bring the system profitably to a lower energy state. There's something called Eulerian perturbation theory, where you subject the equilibrium state to small-amplitude perturbations, formulate those perturbations in the lab frame, and ask whether the perturbations oscillate, grow, or decay. And there's something called Lagrangian perturbation theory, which is same as Eulerian perturbation theory but is formulated in the frame of fluid. Each of these has its advantages depending on the equilibrium state, boundary conditions, and questions being asked. Eulerian perturbation theory is the most straightforward procedure, so we'll start there.

### IV.1. A primer on instability

Before attacking the MHD equations, though, let's do something simpler to establish notation and learn the procedure. Consider the following ordinary differential equation:

$$\frac{d^2 x}{dt^2} + 2\nu \frac{dx}{dt} + \Omega^2(x - x_0) = 0, \quad (\text{IV.3})$$

where  $\nu$  and  $\Omega > 0$  are constants. You may recognize this as the equation for a damped simple harmonic oscillator of natural frequency  $\Omega$  whose velocity along the  $x$  axis is damped at a rate  $2\nu$ . But let's not yet commit to any particular sign of  $\nu$ . First, the equilibrium state. This is easy: the oscillator is at rest at  $x = x_0$ . We now displace the oscillator by a small amount  $\xi$ , so that  $x(t) = x_0 + \xi(t)$ . The equation governing this displacement is

$$\frac{d^2 \xi}{dt^2} + 2\nu \frac{d\xi}{dt} + \Omega^2 \xi = 0. \quad (\text{IV.4})$$

This equation admits solutions  $\xi \sim \exp(-i\omega t)$ , where  $\omega$  is a complex frequency that satisfies the *dispersion relation*

$$\omega^2 + 2i\omega\nu - \Omega^2 = 0 \implies \omega = -i\nu \pm \sqrt{\Omega^2 - \nu^2}. \quad (\text{IV.5})$$

How do we assess stability? If the imaginary part of  $\omega$  is positive, then  $-i\omega$  has a positive real part, and the displacements will grow exponentially in time. If the imaginary part of  $\omega$  is negative, then  $-i\omega$  has a negative real part, and this corresponds to exponential decay of the perturbation. If  $\omega$  additionally has a real part, then this represents a growing or decaying oscillator. It's clear from a cursory glance at the dispersion relation (IV.5) that the perturbations oscillate and decay exponentially if  $\Omega > \nu > 0$ . If  $\nu > \Omega > 0$ , then the perturbations decay without oscillating. But if  $\nu < 0$ , then there is always an exponentially growing solution. Thus,  $\nu > 0$  is the *stability criterion* for this system.

Now, suppose the equation of interest were instead

$$\frac{d^2x}{dt^2} + 2\nu\frac{dx}{dt} + \Omega^2 \sin(x - x_0) = 0. \quad (\text{IV.6})$$

The equilibrium is still the same, but if we want simple harmonic oscillator solutions, we're only going to get them if the displacement is small, i.e.,  $|\xi| \ll x_0$ . In that case, we can Taylor expand  $\sin(x - x_0) \approx \xi - \xi^3/6 + \dots$ . To leading order in  $\xi$ , we're back to where we started with (IV.4). This is *linear theory*: identify an equilibrium, perturb the system about that equilibrium, and drop all terms nonlinear in the perturbation amplitude.

Note that we are not solving an initial value problem. We are agnostic about the initial conditions and only ask whether some disturbance will ultimately grow or decay. In some situations (most notable, Landau damping), solving the initial value problem is absolutely essential to obtain the full solution and all the physics involved. But if you just want to calculate the wave-like response of a system to infinitesimally small perturbations and learn whether such a response grows or decays, you need only adopt solutions  $\sim \exp(-i\omega t)$ , find the dispersion relation for  $\omega$  vs  $\mathbf{k}$ , and examine the sign of its imaginary part. (The difference is related to a Laplace vs a Fourier transform in time.)

## IV.2. Linearized MHD equations

Good, now let's do some MHD. Take (IV.1) and write

$$\rho = \rho_0(\mathbf{r}) + \delta\rho(t, \mathbf{r}), \quad \mathbf{u} = \delta\mathbf{u}(t, \mathbf{r}), \quad P = P_0(\mathbf{r}) + \delta P(t, \mathbf{r}), \quad \mathbf{B} = \mathbf{B}_0(\mathbf{r}) + \delta\mathbf{B}(t, \mathbf{r});$$

i.e., consider a stratified, stationary equilibrium state threaded by a magnetic field and subject it to perturbations. Never mind how the equilibrium is set up – it is what it is, and we'll perturb it. Neglecting all terms quadratic in  $\delta$ , equations (IV.1) become

$$\frac{\partial\delta\rho}{\partial t} = -(\delta\mathbf{u} \cdot \nabla)\rho_0 - \rho_0(\nabla \cdot \delta\mathbf{u}), \quad (\text{IV.7})$$

$$\begin{aligned} \frac{\partial\delta\mathbf{u}}{\partial t} = & -\frac{1}{\rho_0} \nabla \left( \delta P + \frac{\mathbf{B}_0 \cdot \delta\mathbf{B}}{4\pi} \right) + \frac{\delta\rho}{\rho_0^2} \nabla \left( P_0 + \frac{B_0^2}{8\pi} \right) \\ & + \frac{(\mathbf{B}_0 \cdot \nabla)\delta\mathbf{B}}{4\pi\rho_0} + \frac{(\delta\mathbf{B} \cdot \nabla)\mathbf{B}_0}{4\pi\rho_0} - \nabla\delta\Phi, \end{aligned} \quad (\text{IV.8})$$

$$\frac{\partial\delta\mathbf{B}}{\partial t} = -(\delta\mathbf{u} \cdot \nabla)\mathbf{B}_0 + (\mathbf{B}_0 \cdot \nabla)\delta\mathbf{u} - \mathbf{B}_0(\nabla \cdot \delta\mathbf{u}), \quad (\text{IV.9})$$

$$\frac{\partial}{\partial t} \left( \frac{\delta P}{P_0} - \gamma \frac{\delta\rho}{\rho_0} \right) = -\delta\mathbf{u} \cdot \nabla \ln \frac{P_0}{\rho_0}. \quad (\text{IV.10})$$

(A quick way of getting these is to think of  $\delta$  as a differential operator that commutes with partial differentiation.) Pretty much every gradient of an equilibrium quantity here will give an instability! (Otherwise, you just get back simple linear waves on a homogeneous background.) So let's not analyze this all at once. But I write this system of equations here for two important reasons: (i) it makes clear that we can adopt solutions  $\delta \sim \exp(-i\omega t)$  for the perturbations, since the equations are linear in the fluctuation amplitudes; (ii) we can only adopt full plane-wave solutions  $\delta \sim \exp(-i\omega t + i\mathbf{k} \cdot \mathbf{r})$  if the fluctuations vary on length scales much smaller than that over which the background varies (the so-called WKB approximation). Otherwise, we have to worry about the exact structure of the background gradients and their boundary conditions.

So these are the themes of most linear stability analyses: a WKB approximation whereby plane-wave solutions are assumed on top of a background state that is slowly varying, and a focus only on whether fluctuations grow or decay rather than their specific spatio-temporal evolution from a set of initial conditions.

### IV.3. Lagrangian versus Eulerian perturbations

There is one last thing worth discussing before proceeding with a linear stability analysis of the MHD equations. Just as there is an Eulerian time derivative and a Lagrangian time derivative, there is Eulerian perturbation theory and Lagrangian perturbation theory. The former, in which perturbations are denoted by a ' $\delta$ ', measures the change in a quantity at a particular point in space. For example, if the equilibrium density at  $\mathbf{r}$ ,  $\rho(\mathbf{r})$ , is changed at time  $t$  by some disturbance to become  $\rho'(t, \mathbf{r})$ , then we denote the Eulerian perturbation of the density by

$$\rho'(t, \mathbf{r}) - \rho(\mathbf{r}) \doteq \delta\rho \ll \rho(\mathbf{r}). \quad (\text{IV.11})$$

Again, these perturbations are taken *at fixed position*. The latter – Lagrangian perturbation theory – concerns the evolution of small perturbations about a background state *within a particular fluid element* as it undergoes a displacement  $\boldsymbol{\xi}$ . For example, if a particularly fluid element is displaced from its equilibrium position  $\mathbf{r}$  to position  $\mathbf{r} + \boldsymbol{\xi}$ , then the density of that fluid element changes by an amount

$$\rho'(t, \mathbf{r} + \boldsymbol{\xi}) - \rho(\mathbf{r}) \doteq \Delta\rho. \quad (\text{IV.12})$$

This is a Lagrangian perturbation. To linear order,  $\delta$  and  $\Delta$  are related by

$$\Delta\rho \simeq \rho'(t, \mathbf{r}) + \boldsymbol{\xi} \cdot \nabla\rho(\mathbf{r}) - \rho(\mathbf{r}) = \delta\rho + \boldsymbol{\xi} \cdot \nabla\rho. \quad (\text{IV.13})$$

There are many situations in which a Lagrangian approach is easier to use than an Eulerian approach; there are also some situations in which doing so is absolutely necessary (e.g., see §IIIe of Balbus (1988) and §Ic of Balbus & Soker (1989) for discussions of the perils of using Eulerian perturbations in the context of local thermal instability).

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*Question:* It is possible to have zero Eulerian perturbation and yet have finite Lagrangian perturbation. What does this mean physically? Is there a physical change in the system?

---

The Lagrangian velocity perturbation  $\Delta\mathbf{u}$  is given by

$$\Delta\mathbf{u} \doteq \frac{D\boldsymbol{\xi}}{Dt} = \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \boldsymbol{\xi}, \quad (\text{IV.14})$$

where  $\mathbf{u}$  is the background velocity. It is the instantaneous time rate of rate of the



displacement of a fluid element, taken relative to the unperturbed flow. Because  $\Delta \mathbf{u} = \delta \mathbf{u} + \boldsymbol{\xi} \cdot \nabla \mathbf{u}$ , we have

$$\delta \mathbf{u} = \frac{\partial \boldsymbol{\xi}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \mathbf{u}. \quad (\text{IV.15})$$

Note the additional  $\boldsymbol{\xi} \cdot \nabla \mathbf{u}$  term, representing a measurement of the background fluid gradients by the fluid displacement.

*Exercise.* Let  $\mathbf{u} = R\Omega(R)\hat{\varphi}$ , as in a differentially rotating disk in cylindrical coordinates. Consider a displacement  $\boldsymbol{\xi}$  with radial and azimuthal components  $\xi_R$  and  $\xi_\varphi$ , each depending upon  $R$  and  $\varphi$ . Show that

$$\frac{D\xi_R}{Dt} = \delta u_R \quad \text{and} \quad \frac{D\xi_\varphi}{Dt} = \delta u_\varphi + \xi_R \frac{d\Omega}{d \ln R}. \quad (\text{IV.16})$$

The second term in the latter equation accounts for the stretching of radial displacements into the azimuthal direction by the differential rotation.

You can think of  $\delta$  and  $\Delta$  as difference operators, since we're only working to linear order in the perturbation amplitude: e.g.,

$$\delta \left( \frac{1}{\rho} \right) = \frac{1}{\rho + \delta\rho} - \frac{1}{\rho} \simeq -\frac{\delta\rho}{\rho^2}.$$

But you must be very careful when mixing Eulerian and Lagrangian points of view. Prove the following commutation relations:

$$\begin{aligned} (i) \quad & \left[ \delta, \frac{\partial}{\partial t} \right] = 0; \\ (ii) \quad & \left[ \delta, \frac{\partial}{\partial x_i} \right] = 0; \\ (iii) \quad & \left[ \Delta, \frac{\partial}{\partial t} \right] = -\frac{\partial \xi_j}{\partial t} \frac{\partial}{\partial \xi_j}; \\ (iv) \quad & \left[ \Delta, \frac{\partial}{\partial x_i} \right] = -\frac{\partial \xi_j}{\partial x_i} \frac{\partial}{\partial \xi_j}; \\ (v) \quad & \left[ \Delta, \frac{D}{Dt} \right] = 0; \\ (vi) \quad & \left[ \Delta, \frac{D}{Dx_i} \right] = -\xi_j \frac{\partial}{\partial x_j} \frac{D}{Dt}; \\ (vii) \quad & \left[ \frac{\partial}{\partial x_i}, \frac{D}{Dt} \right] = \frac{\partial u_j}{\partial x_i} \frac{\partial}{\partial x_j}. \end{aligned}$$

You can use these to show that the linearized continuity equation, induction equation, and internal energy equation are

$$\frac{\Delta \rho}{\rho} = -\nabla \cdot \boldsymbol{\xi}, \quad (\text{IV.17})$$

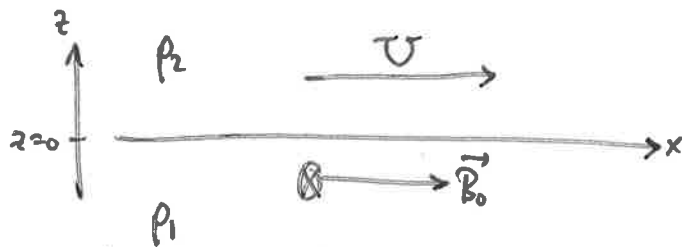
$$\Delta \mathbf{B} = \mathbf{B} \cdot \nabla \boldsymbol{\xi} - \mathbf{B} \nabla \cdot \boldsymbol{\xi}, \quad (\text{IV.18})$$

$$\frac{\Delta T}{T} = -(\gamma - 1) \nabla \cdot \boldsymbol{\xi}, \quad (\text{IV.19})$$

respectively. These forms are particularly useful for linear analyses.

## SOME MHD INSTABILITIES

- Kelvin-Helmholtz instability (KHI). Using Lagrangian perturbation, this is quite easy. Consider two uniform fluids separated by a discontinuous interface at  $z=0$ . The fluid above the interface ( $z>0$ ) has density  $\rho_2$  and equilibrium velocity  $\vec{u}_0 = U \hat{x}$ . The fluid below the interface ( $z<0$ ) has density  $\rho_1$  and is stationary. (We can always transform to a frame in which this fluid is stationary, so why not take advantage of that?) There is a uniform magnetic field  $\vec{B}_0 = B_{0x} \hat{x} + B_{0y} \hat{y}$  oriented parallel to the interface that permeates all of the fluid, which we take to be perfectly conducting. Assume the fluid is incompressible.



We seek the dispersion relation governing small-amplitude perturbations. Take the momentum equation and apply  $\delta$ :

$$\delta \left[ \rho \frac{d\vec{u}}{dt} = -\vec{\nabla} \left( p + \frac{B^2}{8\pi} \right) + \frac{\vec{B} \cdot \vec{\nabla} \vec{B}}{4\pi} \right]$$

$$\Rightarrow \rho \frac{d^2 \vec{\xi}}{dt^2} = -\vec{\nabla} \delta \left( p + \frac{B^2}{8\pi} \right) + \frac{\vec{B}_0 \cdot \vec{\nabla} \delta \vec{B}}{4\pi},$$

since  $\vec{\nabla} B_0 = \vec{\nabla} p_0 = 0$ . Use linearized induction equation,

$\delta \vec{B} = (\vec{B}_0 \cdot \vec{\nabla}) \vec{\xi}$  (since  $\vec{\nabla} B_0 = 0$  and  $\vec{\nabla} \cdot \vec{u} = \vec{\nabla} \cdot \vec{\xi} = 0$ ), to obtain

$$\textcircled{*} \left[ \frac{d^2}{dt^2} - \frac{(\vec{B}_0 \cdot \vec{\nabla})^2}{4\pi\rho} \right] \vec{\xi} = -\frac{1}{\rho} \vec{\nabla} \delta \left( p + \frac{B^2}{8\pi} \right) \equiv -\frac{1}{\rho} \vec{\nabla} \delta \pi.$$

Take  $\nabla \cdot$  of this:

$$\left[ \frac{d^2}{dt^2} - \frac{(\vec{B}_0 \cdot \nabla)^2}{4\pi\rho} \right] \nabla \cdot \vec{E} = -\frac{1}{\rho} \nabla^2 \delta\pi = 0 \text{ by incompressibility}$$

$$\Rightarrow \nabla^2 \delta\pi = \left( -k_x^2 - k_y^2 + \frac{\partial^2}{\partial z^2} \right) \delta\pi = 0 \Rightarrow \delta\pi \propto \exp\left( ik_x x + ik_y y - |kz| \right)$$

where  $k = \sqrt{k_x^2 + k_y^2}$ . Adopt solutions  $\xi \sim e^{-i\omega t}$ . Then equation  $\otimes$  above and below the interface becomes

$$\left[ (-i\omega + ik_x U)^2 + (\vec{k} \cdot \vec{V}_A)_2^2 \right] \xi_{z2} = + \frac{1}{\rho_2} |k| \delta\pi_2,$$

$$\left[ (-i\omega)^2 + (\vec{k} \cdot \vec{V}_A)_1^2 \right] \xi_{z1} = - \frac{1}{\rho_1} |k| \delta\pi_1,$$

respectively. At the interface,  $\xi_{z1} = \xi_{z2}$  and  $\delta\pi_1 = \delta\pi_2$ .

Because  $\nabla \rho_0 = 0$ , the latter implies  $\delta\pi_1 = \delta\pi_2$ . Matching these, we find

$$(\omega - k_x U)^2 \rho_2 + \omega^2 \rho_1 = \frac{(\vec{k} \cdot \vec{B}_0)^2}{2\pi}$$

$$\Rightarrow \omega = k_x U \frac{\rho_2}{\rho_1 + \rho_2} \left\{ 1 \pm \left( \frac{\rho_1}{\rho_2} \right)^{1/2} \left[ \frac{(\vec{k} \cdot \vec{B}_0)^2}{\pi \bar{\rho} k_x^2 U^2} - 1 \right]^{1/2} \right\}$$

where  $\bar{\rho} \equiv \frac{2\rho_1 \rho_2}{\rho_1 + \rho_2}$  is the reduced mass density. For

stability, the discriminant must be  $> 0$ ; thus,

$$\boxed{\frac{(\vec{k} \cdot \vec{B}_0)^2}{4\pi\bar{\rho}} > \left(\frac{k_x U}{2}\right)^2}$$

magn. tension > velocity shear

Kelvin-Helmholtz stability.

Note that, if  $B_{0x} = 0$ , then this reads  $\left(\frac{k_y}{k_x}\right)^2 \frac{B_{0y}^2}{4\pi\bar{\rho}} > \left(\frac{U}{2}\right)^2$ , which is always unstable for small enough  $|k_y/k_x|$ , no matter how strong is  $B_{0y}$ .

The physics: an upwardly displaced distortion of the interface into region 2 causes a constriction of the velocity there, and the fluid must move faster to conserve its mass. But when it moves faster, the pressure must drop (Bernoulli!). The opposite happens below the interface. Now there is a pressure gradient pushing upwards, reinforcing the displacement, and the process runs away. That's why pressure perturbations were vital in (\*).

Question: Does this instability occur in a linear shear flow? (i.e.,  $\vec{u}_0 = S z \hat{x}$ ) No! Drop the magnetic field for simplicity. With  $\vec{u}_0 = u_0(z) \hat{x}$ , one can show from (using  $\vec{\nabla} \cdot \vec{u} = 0$ )

$$\left. \begin{aligned} ik_x \delta u_x + \delta u_z' &= 0 \\ -i(\omega - k_x u_0) \delta u_x + \delta u_z u_0' &= -ik_x \delta p / \rho \\ -i(\omega - k_x u_0) \delta u_z &= -\delta p' / \rho \end{aligned} \right\} \delta u_z'' - k_x^2 \delta u_z = \frac{k_x u_0'' \delta u_z}{\omega - k_x u_0}. \text{ Now,}$$

Multiply this by  $\delta u_z^*$  ( $\leftarrow$  c.c.) and integrate between upper and lower boundaries  $\pm L$ .

$$\int_{-L}^L dz \left( \delta u_z^* \delta u_z'' - k_x^2 |\delta u_z|^2 \right) = \int_{-L}^L dz \frac{k_x u_0'' |\delta u_z|^2}{\omega - k_x u_0}$$

$$= \delta u_z^* \delta u_z' \Big|_{-L}^{+L} - \int_{-L}^L dz |\delta u_z'|^2$$

$\downarrow$   
 $= 0$  if periodic,  
 or if  $\delta u_z$  or  $\delta u_z'$   
 vanishes at boundaries

$$\Rightarrow - \int_{-L}^L dz \left[ |\delta u_z'|^2 + k_x^2 |\delta u_z|^2 + \frac{k_x u_0'' |\delta u_z|^2}{\omega - k_x u_0} \right] = 0$$

If the system is unstable,  $\omega$  must have an imaginary part,  $\omega_I$ . Writing  $\omega = \omega_R + i\omega_I$ , the imaginary part of the above equation is

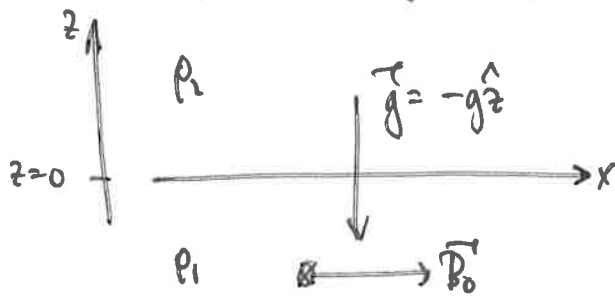
$$\omega_I k_x \int_{-L}^L dz \left( \frac{|\delta u_z|^2 u_0''}{|\omega - k_x u_0|^2} \right) = 0$$

$\Rightarrow u_0''$  must be positive over part of the integration range, and negative over the rest; i.e.,  $u_0''$  must pass through zero.

Thus, instability requires an inflection point!

(J.W.S. Rayleigh, Proc. London Math. Soc. 9, 57 (1880))

- Rayleigh-Taylor instability (RTI). Consider two fluids separated by a discontinuous interface at  $z=0$  in the presence of a constant gravitational field  $\vec{g} = -g\hat{z}$ . The fluid above the interface ( $z>0$ ) has uniform density  $\rho_2$ ; the fluid below the interface ( $z<0$ ) has uniform density  $\rho_1$ . Both fluids are initially stationary. There is a uniform magnetic field  $\vec{B}_0 = B_{0x}\hat{x} + B_{0y}\hat{y}$  oriented parallel to the interface that permeates all of the fluid, which we take to be perfectly conducting. Again, assume  $\vec{\nabla} \cdot \vec{u} = 0$ .



Either side of the interface,  $-\frac{1}{\rho} \frac{dp}{dz} = g$

$$\Rightarrow g = -\frac{1}{\rho_1} \frac{dp_1}{dz} = -\frac{1}{\rho_2} \frac{dp_2}{dz}$$

As before,

$$\delta \left[ \rho \frac{d\vec{u}}{dt} = -\vec{\nabla} \left( p + \frac{B^2}{8\pi} \right) + \frac{\vec{B} \cdot \vec{\nabla} \vec{B}}{4\pi} + \rho \vec{g} \right]$$

$$\Rightarrow \rho \frac{d^2 \vec{\xi}}{dt^2} = -\vec{\nabla} \delta \left( p + \frac{B^2}{8\pi} \right) + \frac{\vec{B}_0 \cdot \vec{\nabla} \delta \vec{B}}{4\pi} - \vec{\xi} \cdot \vec{\nabla} (\vec{\nabla} p) + \delta \rho \vec{g}$$

$$= (\delta p - \vec{\xi} \cdot \vec{\nabla} p) \vec{g}$$

$$= \delta p \vec{g} = 0 \text{ (incomp.)}$$

Following the KHI calculation, this gives

$$\left[ -\omega^2 + \frac{(\vec{k} \cdot \vec{v}_{A0})_2^2}{2\pi} \right] \xi_{z2} = + \frac{1}{\rho_2} |k| \delta\pi_2,$$

$$\left[ -\omega^2 + \frac{(\vec{k} \cdot \vec{v}_{A0})_1^2}{2\pi} \right] \xi_{z1} = - \frac{1}{\rho_1} |k| \delta\pi_1.$$

likewise,  $\xi_{z1} = \xi_{z2}$  and  $\delta\pi_1 = \delta\pi_2$  at the interface.  
But because  $\nabla\rho_0 \neq 0$ , we have

$$\delta\pi_1 = \delta\pi_2 \Rightarrow \delta\pi_1 - \xi_{z1} \rho_1 g = \delta\pi_2 - \xi_{z2} \rho_2 g$$

$$\xi_{z1} = \xi_{z2} \Rightarrow \delta\pi_1 - \delta\pi_2 = (\rho_1 - \rho_2) \xi_{z2} g.$$

$$\begin{aligned} \Rightarrow -\rho_1 \left[ -\omega^2 + \frac{(\vec{k} \cdot \vec{v}_{A0})_1^2}{2\pi} \right] \xi_z - \rho_2 \left[ -\omega^2 + \frac{(\vec{k} \cdot \vec{v}_{A0})_2^2}{2\pi} \right] \xi_z \\ = |k| (\rho_1 - \rho_2) \xi_z g, \end{aligned}$$

or

$$\omega^2 (\rho_1 + \rho_2) - \frac{(\vec{k} \cdot \vec{B}_0)^2}{2\pi} = |k| (\rho_1 - \rho_2) g.$$

$$\Rightarrow \omega = \pm \left[ \frac{(\vec{k} \cdot \vec{B}_0)^2}{2\pi(\rho_1 + \rho_2)} + |k| g \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right]^{1/2}$$

For stability, the discriminant must be  $> 0$ ; thus,

$$\left[ \rho_1 > \rho_2 - \frac{(\vec{k} \cdot \vec{B}_0)^2}{2\pi |k| g} \right] \text{ Rayleigh-Taylor stability}$$

For instability, the fluid on top must be heavy enough for the gravitational force acting on a density fluctuation of wavelength  $2\pi/|k|$  to overcome the stabilizing magnetic tension. (Note that, if  $\vec{B}_0$  is not oriented along the interface, no amount of magnetic field can stabilize the system.)



- Convective instability. The RTI is but one example of convective instability; here we'll generalize the calculation for a continuously stratified MHD fluid. We will still have  $-\frac{1}{\rho} \frac{d\rho}{dz} = g = \text{constant}$  as our equilibrium state, but will allow perturbations throughout the fluid:

$$\rho = \rho_0(z) + \delta\rho, \quad p = p_0(z) + \delta p, \quad \vec{u} = \vec{v} + \delta\vec{u}.$$

For simplicity, let's ignore the magnetic field — it'll simply act to stabilize the system via its tension (as long as  $B_0 = \text{constant}$ ). The linearized eqns. are then

$$\frac{\partial}{\partial t} \delta\rho + \rho_0(z) \vec{\nabla} \cdot \delta\vec{u} + \delta\vec{u} \cdot \vec{\nabla} \rho_0 = 0$$

$$\rho_0 \frac{\partial}{\partial t} \delta\vec{u} = -\vec{\nabla} \delta p - \delta\rho g \hat{z}$$

$$\frac{\partial}{\partial t} \left( \frac{\delta p}{\rho_0} - \gamma \frac{\delta\rho}{\rho_0} \right) + \delta\vec{u} \cdot \vec{\nabla} \ln \rho_0 \rho_0^{-\gamma} = 0$$

Solutions to this set of eqns. are  $\propto e^{-i\omega t}$ :

$$-i\omega \frac{\delta\rho}{\rho_0} + \vec{\nabla} \cdot \delta\vec{u} + \delta u_z \frac{d \ln \rho_0}{dz} = 0,$$

$$-i\omega \delta\vec{u} = -\frac{1}{\rho_0} \vec{\nabla} \delta p - \frac{\delta\rho}{\rho_0} g \hat{z},$$

$$-i\omega \left( \frac{\delta p}{\rho_0} - \gamma \frac{\delta\rho}{\rho_0} \right) + \delta u_z \frac{d \ln \rho_0 \rho_0^{-\gamma}}{dz} = 0.$$



In general, we cannot Fourier transform these eqns. in  $z$ , because the coefficients in front of the perturbed quantities are  $z$ -dependent. But we can do so in the horizontal (say,  $x$ ) direction:

$$-i\omega \frac{\delta p}{\rho_0} + ik_x \delta u_x + \frac{d\delta u_z}{dz} + \delta u_z \frac{d \ln \rho_0}{dz} = 0,$$

$$-i\omega \delta u_x = -ik_x \frac{\delta p}{\rho_0},$$

$$-i\omega \delta u_z = -\frac{1}{\rho_0} \frac{d\delta p}{dz} - \frac{\delta p}{\rho_0} g,$$

$$-i\omega \left( \frac{\delta p}{\rho_0} - \gamma \frac{\delta p}{\rho_0} \right) + \delta u_z \frac{d \ln \rho_0 \rho_0^{-\gamma}}{dz} = 0,$$

where now the fluctuations are  $z$ -dependent Fourier amplitudes. Denoting  $\delta u = -i\omega \xi$ , and dropping the equilibrium "0" subscripts for notational ease, we have

$$\textcircled{A} \quad \frac{\delta p}{\rho} + ik_x \xi_x + \xi_z' + \xi_z \frac{d \ln \rho}{dz} = 0,$$

$$\textcircled{B} \quad -\omega^2 \xi_x = -ik_x \frac{\delta p}{\rho},$$

$$\textcircled{C} \quad -\omega^2 \xi_z = -\frac{1}{\rho} \frac{d\delta p}{dz} - \frac{\delta p}{\rho} g,$$

$$\textcircled{D} \quad \frac{\delta p}{\rho} = \gamma \frac{\delta p}{\rho} - \xi_z \frac{d \ln \rho \rho^{-\gamma}}{dz}.$$

$$\textcircled{B} \text{ and } \textcircled{D} \Rightarrow -\omega^2 \xi_x = -ik_x \frac{P}{\rho} \left[ \gamma \frac{\delta p}{\rho} - \xi_z \frac{d \ln P}{dz} P^{-\gamma} \right]$$

$$\text{and } \textcircled{A} \Rightarrow -\omega^2 \xi_x = +ik_x \frac{P}{\rho} \gamma \left[ +ik_x \xi_x + \xi_z' + \xi_z \frac{d \ln P}{dz} \right] + ik_x \frac{P}{\rho} \xi_z \frac{d \ln P}{dz} P^{-\gamma}$$

$$\Rightarrow (-\omega^2 + k_x^2 a^2) \xi_x = ik_x a^2 \xi_z' + \frac{ik_x a^2}{\gamma} \frac{d \ln P}{dz} \xi_z,$$

where  $a^2 \equiv \gamma P / \rho$ . Note:  $g = -\frac{a^2}{\gamma} \frac{d \ln P}{dz}$ , so this is

$$\textcircled{A} \left[ (-\omega^2 + k_x^2 a^2) \xi_x = ik_x a^2 \xi_z' - ik_x \xi_z g \right]$$

$$\textcircled{A} \Rightarrow \frac{\delta p}{\rho} = -\xi_z' - \xi_z \frac{d \ln P}{dz} - \frac{ik_x [ik_x a^2 \xi_z' - ik_x \xi_z g]}{-\omega^2 + k_x^2 a^2}$$

$$\Rightarrow \left[ \frac{\delta p}{\rho} = \frac{\omega^2 \xi_z' + \left[ (\omega^2 - k_x^2 a^2) \frac{d \ln P}{dz} - k_x^2 g \right] \xi_z}{k_x^2 a^2 - \omega^2} \right] \textcircled{C}$$

$$\Rightarrow \left[ \frac{\delta p}{\rho} = \frac{1}{k_x^2 a^2 - \omega^2} \left[ \gamma \omega^2 \xi_z' + \gamma \left( (\omega^2 - k_x^2 a^2) \frac{d \ln P}{dz} - k_x^2 g \right) \xi_z + (\omega^2 - k_x^2 a^2) \frac{d \ln P}{dz} P^{-\gamma} \xi_z \right] \right]$$

$$= \frac{1}{k_x^2 a^2 - \omega^2} \left[ \gamma \omega^2 \xi_z' + \omega^2 \frac{d \ln P}{dz} \xi_z \right]$$

$$= \frac{\omega^2}{k_x^2 a^2 - \omega^2} \left[ \gamma \xi_z' + \frac{d \ln P}{dz} \xi_z \right] \textcircled{D}$$

into (c):

$$+ \omega^2 \xi_z = +g \int \frac{\omega^2 \xi_z' + (\omega^2 - k_x^2 a^2) \frac{d \ln p}{dz} \xi_z - k_x^2 g \xi_z}{k_x^2 a^2 - \omega^2}$$

$$+ \frac{1}{\rho} \frac{d}{dz} \left[ \frac{\omega^2 \rho}{k_x^2 a^2 - \omega^2} \left( \xi_z' \gamma + \xi_z \frac{d \ln p}{dz} \right) \right]$$

$$= \frac{\rho \frac{d}{dz} \left( \xi_z' \gamma + \xi_z \frac{d \ln p}{dz} \right)}{k_x^2 a^2 - \omega^2} + \frac{\rho \omega^2}{k_x^2 a^2 - \omega^2} \left( \gamma \xi_z'' + \xi_z' \frac{d \ln p}{dz} + \xi_z \frac{d^2 \ln p}{dz^2} \right)$$

$$- \frac{\omega^2 \rho}{\rho} \frac{k_x^2}{(k_x^2 a^2 - \omega^2)^2} \frac{da^2}{dz} \left( \xi_z' \gamma + \xi_z \frac{d \ln p}{dz} \right)$$

$$\Rightarrow \omega^2 \xi_z = \frac{g}{k_x^2 a^2 - \omega^2} \left[ \omega^2 \xi_z' - k_x^2 g \xi_z + (\omega^2 - k_x^2 a^2) \frac{d \ln p}{dz} \xi_z \right]$$

$$+ \frac{\omega^2}{k_x^2 a^2 - \omega^2} (-g) \left[ \xi_z' \gamma + \xi_z \frac{d \ln p}{dz} \right]$$

$$+ \left( \frac{a^2}{\gamma} \right) \frac{\omega^2}{k_x^2 a^2 - \omega^2} \left[ \xi_z'' \gamma + \xi_z' \frac{d \ln p}{dz} + \xi_z \frac{g \gamma}{a^2} \frac{d \ln p}{dz} \right]$$

$$- \frac{\omega^2}{\gamma} \frac{a^2 k_x^2 a^2}{(k_x^2 a^2 - \omega^2)^2} \frac{d \ln p}{dz} \left[ \xi_z' \gamma + \xi_z \frac{d \ln p}{dz} \right]$$

Multiply by  $\frac{k_x^2 a^2 - \omega^2}{\omega^2 a^2}$  and group:

$$b_{z'}^{\parallel}: 1.$$


$$\begin{aligned}
 b_{z'}^{\perp}: & \frac{g}{a^2} - \frac{g\gamma}{a^2} + \frac{1}{\gamma} \frac{d\ln p}{dz} - \frac{k_x^2 a^2}{\gamma} \frac{1}{(k_x^2 a^2 - \omega^2)} \frac{d\ln T}{dz} \\
 & = \frac{d\ln p}{dz} - \left( \frac{k_x^2 a^2}{k_x^2 a^2 - \omega^2} \right) \frac{d\ln T}{dz} = \frac{d\ln p/dz}{k_x^2 a^2 - \omega^2} \left( k_x^2 a^2 - \omega^2 - k_x^2 a^2 \frac{d\ln T}{d\ln p} \right) \\
 & = \frac{d\ln p/dz}{k_x^2 a^2 - \omega^2} \left( -\omega^2 + k_x^2 a^2 \frac{d\ln p}{d\ln p} \right) = \frac{\omega^2 \frac{d\ln p}{dz} - k_x^2 a^2 \frac{d\ln p}{dz}}{\omega^2 - k_x^2 a^2}
 \end{aligned}$$

$$\begin{aligned}
 b_{z'}^{\perp}: & - \frac{(k_x^2 a^2 - \omega^2)}{a^2} - \frac{k_x^2 g}{\omega^2 a^2} - g \frac{d\ln p}{dz} \frac{k_x^2 a^2 - \omega^2}{\omega^2 a^2} \\
 & - \frac{g}{a^2} \frac{d\ln p}{dz} + \frac{a^2}{\gamma} \frac{1}{a^2} \frac{g\gamma}{a^2} \frac{d\ln T}{dz} - \frac{k_x^2 a^2}{\gamma (k_x^2 a^2 - \omega^2)} \frac{d\ln T}{dz} \frac{d\ln p}{dz} \\
 & = \frac{-1}{k_x^2 a^2 - \omega^2} \left[ \frac{k_x^2 a^2}{\gamma} \frac{d\ln T}{dz} \frac{d\ln p}{dz} + \frac{g k_x^2}{\omega^2} \frac{d\ln p}{dz} (k_x^2 a^2 - \omega^2) \right. \\
 & \quad \left. + \frac{k_x^2 g}{a^2 \omega^2} (k_x^2 a^2 - \omega^2) + \frac{(k_x^2 a^2 - \omega^2)^2}{a^2} \right] \\
 & = \frac{1}{\omega^2 - k_x^2 a^2} \left[ \frac{\omega^4}{a^2} - 2\omega^2 k_x^2 + k_x^4 a^2 - \frac{k_x^2 g^2}{a^2} - \frac{g k_x^2}{\omega^2} \frac{d\ln p}{dz} \right. \\
 & \quad \left. + \frac{k_x^2 a^2}{\gamma} \frac{d\ln T}{dz} \frac{d\ln p}{dz} + \frac{g k_x^4 a^2}{\omega^2} \frac{d\ln p}{dz} + \frac{k_x^4 a^2 g^2}{a^2 \omega^2} \right] \\
 & \quad \left[ -k_x^2 g \left( \frac{d\ln p}{dz} - \frac{d\ln p}{dz} \right) \right]
 \end{aligned}$$

So,  $\xi_2'' + \xi_2' \left[ \frac{\omega^2 \frac{dnp}{dz} - k_x^2 a^2 \frac{dnp}{dz}}{\omega^2 - k_x^2 a^2} \right]$

+  $\xi_2 \left[ \frac{1}{\omega^2 - k_x^2 a^2} \left[ \frac{\omega^4}{a^2} - 2\omega^2 k_x^2 + k_x^4 a^2 - k_x^2 g \frac{dnp}{dz} \left(1 - \frac{1}{\gamma}\right) \right. \right.$

$\left. \left. + g \frac{k_x^4 a^2}{\omega^2} \frac{dnp}{dz} + \frac{k_x^4 g^2}{\omega^2} \right] \right] = 0.$



This is UGLY!!! And we can't solve it analytically anyhow. It's just a stratified fluid — why is it so complicated?! The reason is twofold: (1) this equation mixes up buoyancy and sound waves — distinct physical effects; and (2) the sound and buoyancy frequencies are functions of height. Let's fix this by adopting an ordering: let

$$\frac{d\xi_2}{dz} \sim ik_z \xi_2 = ik_z H \left( \frac{\xi_2}{H} \right) \gg \frac{\xi_2}{H},$$

where  $H \equiv \left| \frac{dz}{dnp} \right| \sim \left| \frac{dz}{dnp} \right|$ . In other words, we assume that  $\xi_2$  varies on a scale  $\ll$  the scale of the background. This is a WKB approach. So...

Let  $\epsilon \equiv \frac{1}{k_z H} \ll 1$ . Also,  $k_x \sim k_z$ . Now, we must make a decision about the size of  $\omega$ , by comparing it with

$\frac{a}{H} = \frac{\delta g}{a} = \int \frac{\delta g}{H}$ . There are two choices of interest:

(i)  $\omega \sim a/H$

(ii)  $\omega \sim ka \sim \frac{(a/H)}{\epsilon} \gg \frac{a}{H}$ .

First, write  $\frac{d\psi_2}{dt} = ik_2 \xi_2$  with  $k_2 H \equiv \frac{1}{\epsilon}$ ;  becomes

$$-k_2^2 \xi_2 + ik_2 \xi_2 \left[ \frac{\omega^2 \frac{d\ln \psi}{dt} - k_x^2 a^2 \frac{d\ln \psi}{dt}}{\omega^2 - k_x^2 a^2} \right] + \frac{\xi_2}{\omega^2 - k_x^2 a^2} \left[ \frac{\omega^4}{a^2} - 2\omega^2 k_x^2 + k_x^4 a^2 - k_x^2 g \frac{d\ln \psi}{dt} \left(1 - \frac{1}{\gamma}\right) + g \frac{k_x^4 a^2}{\omega^2} \frac{d\ln \psi}{dt} + \frac{k_x^4 g^2}{\omega^2} \right] = 0.$$

Now, (i)  $\omega \sim a/H$  gives  $k_x^2 a^2 \gg \omega^2$  and so the dominant terms are

$$-k_2^2 \xi_2 + \xi_2 (-k_x^2) - \xi_2 \frac{g k_x^2}{\omega^2} \frac{d\ln \psi}{dt} - \xi_2 \frac{k_x^2 g^2}{a^2 \omega^2} = 0$$

$$\Rightarrow k^2 + \frac{g k_x^2}{\omega^2} \left[ \frac{d\ln \psi}{dt} + \frac{g}{a^2} \right] = 0.$$

$$\frac{d\ln \psi}{dt} - \frac{1}{\gamma} \frac{d\ln \psi}{dt} = -\frac{1}{\gamma} \frac{d\ln \psi}{dt}^{-\gamma}$$

$$\Rightarrow \omega^2 = \frac{k_x^2}{k^2} \frac{g}{\gamma} \frac{d \ln p p^{-\gamma}}{dz}$$

$$= -\frac{k_x^2}{k^2} \frac{1}{\gamma p} \frac{dp}{dz} \frac{d \ln p p^{-\gamma}}{dz} = \frac{k_x^2}{k^2} N^2$$

where  $N^2$  is the square of the Brunt - Väisälä frequency.  
 If  $N^2 > 0$ , these are called internal waves or g-modes.

Note that different wavenumbers have different velocities (i.e., dispersion) and that  $\omega$  depends on the direction

of  $\vec{k}$ :  $\frac{\partial \omega}{\partial \vec{k}} = \frac{\omega}{k^2} \frac{k_z}{k_x} (k_z \hat{x} - k_x \hat{z})$ , so that  $\vec{k} \cdot \frac{\partial \omega}{\partial \vec{k}} = 0$ .

We'll return to the physical cause of these waves later, after the "Boussinesq approximation" is introduced, but, for now, note that  $N^2 < 0$  (i.e., upwardly decreasing entropy) gives instability. Go boil some water and think about it.

(ii)  $\omega \sim ka \gg a/H$ . This gives the following dominant terms:

$$-k_z^2 + \frac{1}{\omega^2 - k_x^2 a^2} \left( \frac{\omega^4}{a^2} - 2\omega^2 k_x^2 + k_x^4 a^2 \right) = 0$$

$(\omega^2 - k_x^2 a^2)^2 / a^2$

$$\Rightarrow \boxed{\omega^2 = (k_x^2 + k_z^2) a^2}$$

Sound waves!



Now, sound waves are often a nuisance in many calculations. They mostly play a rather boring role, and often serve only to make the algebra more tedious. There is something called the Boussinesq approximation, which rigorously filters out sound waves. Let's see how this works in our convection problem...

Return to (A) with  $\omega \sim \epsilon ka$ :

$$\frac{\delta p}{\rho} \approx \frac{\omega^2}{k_x^2 a^2} i k_z \xi_z \gamma \quad \text{if } k_z \neq 0;$$

$$\approx \frac{\omega^2}{k_x^2 a^2} \frac{d \ln p}{dz} \xi_z \quad \text{otherwise.}$$

So, perhaps we should have dropped perturbations to the gas pressure at some point. Where? In the momentum equation? Hmm... careful. Consider (B) with  $\frac{\delta p}{\rho} \approx \frac{\omega^2}{k_x^2 a^2} i k_z \xi_z \gamma$

$\Rightarrow \xi_x = -\xi_z \frac{k_z}{k_x}$ , or  $\mathbf{k} \cdot \xi = 0$ . Looks like pressure fluctuations are enforcing (near) incompressibility. Best not to drop them! And (C)?

$$-\omega^2 \xi_z = -i k_z \frac{\rho}{\rho} \frac{\omega^2}{k_x^2 a^2} i k_z \xi_z \gamma - \frac{\delta p}{\rho} g$$

Same order!

Okay. So, pressure fluctuations are small, but not so small that they can be dropped from the momentum eqn. What about the entropy eqn?

$$\textcircled{D} \Rightarrow \frac{\delta p}{p} = \gamma \frac{\delta p}{p} - \xi_z \frac{d \ln p}{dz} p^{-\gamma}$$

$\uparrow$   $\uparrow$   
 $\sim \frac{\xi_z}{H} \frac{\omega^2}{k_x^2 a^2}$   $\sim \frac{\xi_z}{H}$  required for internal waves

or  $\sim i k_z \xi_z \gamma \frac{\omega^2}{k_x^2 a^2}$ , either way... it's small. So, drop  $\delta p$

from entropy equation! What does that leave us with?

$$\gamma \frac{\delta p}{p} \approx \xi_z \frac{d \ln p}{dz} p^{-\gamma}$$

$$\Rightarrow \frac{\delta p}{p} \approx \frac{\partial \ln p}{\partial z} \frac{\xi_z}{H}. \text{ Ah! Look at } \textcircled{A}: \frac{\delta p}{p} + i k_z \xi_z + i k_x \xi_x$$

$\swarrow$   $\searrow$   $\downarrow$   
 $\sim \frac{\xi_z}{H}$   $\sim k \xi_z$   $+ \xi_z \frac{d \ln p}{dz} = 0$   
 $\sim \frac{\xi_z}{H}$

So, to leading order, we have  $\nabla \cdot \vec{v} = 0$  — incompressibility!  
 Okay, things are consistent, and we have the Boussinesq approx:

$$\frac{\delta p}{\rho} \sim \frac{1}{kH} \frac{\delta p}{\rho} \ll \frac{\delta p}{\rho} \sim \frac{\delta u}{a} \ll \frac{k \delta u}{\omega} \sim k \xi \sim (kH) \frac{\delta u}{a}$$

Or, defining the Mach number  $M$  and taking it to be small ( $\sim \epsilon$ ),

$$\frac{\delta u}{a} \sim \frac{\delta p}{\rho} \sim \frac{\delta T}{T} \sim \frac{1}{M} \frac{\delta p}{\rho} \sim \frac{1}{kH} \sim \epsilon \ll 1.$$

In practice, this means:

- 1) Assume (near) incompressibility ( $\vec{\nabla} \cdot \vec{\delta u} = 0$ )
- 2) Drop  $\delta p$  everywhere EXCEPT the momentum eqn. They are enforcing (near) incompressibility.
- 3) Keep  $\delta p$  everywhere EXCEPT the continuity eqn. They interact with gravity to give buoyancy.

Watch how much simpler this is...

$$\textcircled{A} \rightarrow ik_x \xi_x + ik_z \xi_z = 0$$

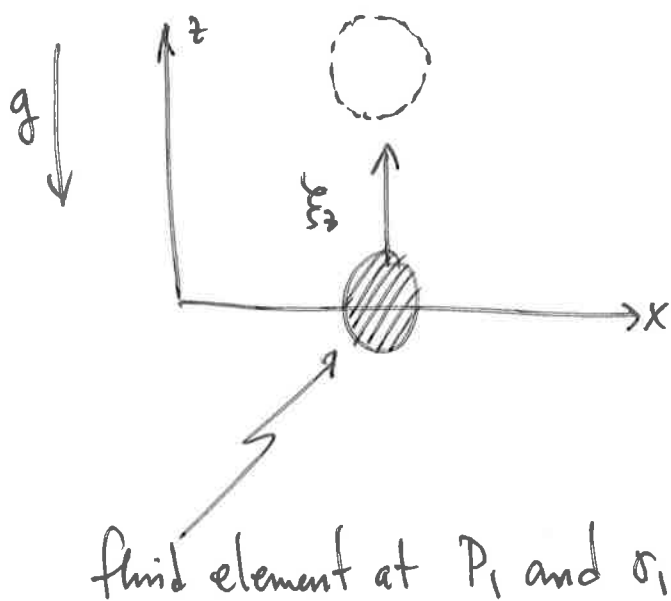
$$\textcircled{B} \rightarrow -\omega^2 \xi_x = -ik_x \frac{\delta p}{\rho} \quad \rightarrow \quad \frac{\delta p}{\rho} = \frac{i\omega^2 k_z}{k_x^2} \xi_z$$

$$\textcircled{C} \rightarrow -\omega^2 \xi_z = -ik_z \frac{\delta p}{\rho} - \frac{\delta p}{\rho} g$$

$$\textcircled{D} \rightarrow 0 = \gamma \frac{\delta p}{\rho} - \xi_z \frac{d \ln \rho}{dz} \quad \rightarrow \quad \omega^2 \xi_z = \frac{k_x^2}{k_z^2} g \frac{\delta p}{\rho}$$

$$\omega^2 = \frac{k_x^2}{k_z^2} N^2. \text{ Done!}$$

What we've done here is eliminated the restoring pressure forces that drive sound waves, essentially by assuming that  $a^2$  is so large that sound waves propagate instantaneously. When the restoring force is purely external (e.g., gravity), the flow behaves as though it were incompressible (nearly). Physically, a slow-moving fluid element remains in pressure balance with its surroundings. This readjustment is what makes buoyancy waves and convection possible. Let us see that explicitly.



$$P_2 < P_1$$

$$\sigma_2 > \sigma_1$$

$$P_1$$

$$\sigma_1$$

where  $\sigma \equiv P \rho^{-\gamma}$  is the entropy variable.

fluid element at  $P_1$  and  $\sigma_1$

Displace fluid element upwards while conserving its entropy. Now it has less entropy than its surroundings. With pressure balance holding, this means that it is also denser than its surroundings. It must fall back to its equilibrium position. Overshooting, it will oscillate at frequency  $N$ . (Mathematically,  $\Delta \sigma = 0 \Rightarrow \Delta p / \rho = \gamma \Delta p / \rho \Rightarrow \vec{\zeta} \cdot \vec{\nabla} \ln p = \gamma \frac{\delta p}{\rho} + \gamma \vec{\zeta} \cdot \vec{\nabla} \ln p \Rightarrow \frac{\delta p}{\rho} = \frac{N^2}{g} \zeta_z$ .)

Now, consider  $\sigma_2 < \sigma_1$ . Our upwardly displaced fluid element has more entropy than its surroundings, and it will continue to rise  $\rightarrow$  convective instability. The (Karl) Schwarzschild criterion for convective stability is  $\boxed{N^2 > 0}$ .

Bonus: Exact solution to  for an isothermal atmosphere.

Suppose  $\frac{d\rho}{dz} = \frac{d\rho}{dz}$  ( $T = \text{const}$ ) Then  $\frac{d\rho}{dz} = -\frac{\rho g}{a^2} = \text{const}$ .

$\Rightarrow \rho = \rho_0 \exp(-z/H)$  with  $H = a^2/\rho g$ . Then we have

$$\xi_z'' - \frac{\xi_z'}{H} + \left[ \frac{\omega^2 - k_x^2 a^2}{a^2} + \frac{k_x g}{H} \left(1 - \frac{1}{\gamma}\right) \frac{1}{\omega^2} \right] \xi_z = 0.$$

let  $\xi_z = f(z) \exp\left(\frac{z}{2H}\right)$ . Then  $\xi_z' = f' e^{z/2H} + \frac{f}{2H} e^{z/2H}$   
 $= f' e^{z/2H} + \frac{\xi_z}{2H}$

$$\xi_z'' = f'' e^{z/2H} + \frac{f'}{H} e^{z/2H} + \frac{\xi_z}{(2H)^2}$$

$$f'' + \frac{f'}{H} + \frac{f}{(2H)^2} - \frac{f'}{H} - \frac{f}{2H^2} + \left[ \dots \right] f = 0.$$

$$f'' + \left[ -\frac{1}{4H^2} + \frac{\omega^2 - k_x^2 a^2}{a^2} + \frac{k_x g}{2H^2} \left(1 - \frac{1}{\gamma}\right) \frac{1}{\omega^2} \right] f = 0.$$

$= \text{const.} \Rightarrow f = \exp(\pm i k_z z)$  with  $k_z^2 =$  ← bracket

$$\Rightarrow -k_z^2 - \frac{1}{4H^2} + \frac{\omega^2 - k_x^2 a^2}{a^2} + \frac{k_x^2 a^2}{\omega^2 H^2} \left( \frac{\gamma-1}{\gamma^2} \right) = 0.$$

Mult. by  $\omega^2 a^2$  and regroup terms:

$$\omega^4 + \omega^2 \left[ -k_z^2 a^2 - \frac{a^2}{4H^2} - k_x^2 a^2 \right] + k_x^2 a^2 \left( \frac{a^2}{H^2} \right) \left( \frac{\gamma-1}{\gamma^2} \right) = 0.$$

$$\Rightarrow \omega^2 = \frac{k_x^2 a^2 + \frac{a^2}{4H^2}}{2} \pm \frac{1}{2} \left[ \left( k_x^2 a^2 + \frac{a^2}{4H^2} \right)^2 - 4k_x^2 a^2 \left( \frac{a^2}{H^2} \right) \left( \frac{\gamma-1}{\gamma^2} \right) \right]^{1/2}.$$

Note that  $N^2 \equiv \frac{g}{\gamma} \frac{d \ln P}{dz} e^{-\gamma} = \left( \frac{1-\gamma}{\gamma} \right) g \frac{d \ln P}{dz} = \frac{a^2}{H^2} \left( \frac{\gamma-1}{\gamma^2} \right).$

So,

$$\omega^2 = \frac{k_x^2 a^2 + \frac{\gamma^2}{\gamma-1} N^2}{2} \pm \frac{1}{2} \left[ \left( k_x^2 a^2 + \frac{\gamma^2 N^2}{\gamma-1} \right)^2 - 4k_x^2 a^2 N^2 \right]^{1/2}$$

If  $(kH)^2 \gg 1$ , this becomes  $\omega^2 = k^2 a^2$  for the sound wave (plus sign) and  $\omega^2 = \frac{k_x^2}{k^2} N^2$  for the g-mode (minus sign). The final term in the square root captures the coupling between these modes.

- A related problem is the Parker Instability, or "magnetic Rayleigh-Taylor instability" (although it is different in detail from RTI and is closer to Schwarzschild Convection). Consider an atmosphere similar to that in our convective instability calculation, but with a magnetic field oriented perpendicularly to gravity with a  $z$ -dependence:  $\vec{B}_0 = B_0(z) \hat{x}$ . The force balance in the equilibrium state now includes a contribution from the magnetic pressure:  $g = -\frac{1}{\rho} \left( \frac{dP}{dz} + \frac{dB_0^2}{dz} \right) = \text{const.}$ , or

$$\frac{g}{a^2} = -\frac{1}{\gamma} \frac{d \ln P}{dz} - \frac{V_{A0}^2}{a^2} \frac{d \ln B_0}{dz}$$

Our equations are almost the same:

$$\frac{\delta p}{\rho} + ik_x \xi_x + \xi_z' + \xi_z \frac{d \ln \rho}{dz} = 0,$$

$$-\omega^2 \xi_x = -ik_x \left( \frac{\delta p}{\rho} + \frac{B_0 \delta B_x}{4\pi \rho} \right) + \frac{ik_x B_0}{4\pi \rho} \delta B_x + \frac{\delta B_z}{4\pi \rho} \frac{dB_0}{dz},$$

$$-\omega^2 \xi_z = -\frac{1}{\rho} \frac{d}{dz} \left( \delta p + \frac{B_0 \delta B_x}{4\pi} \right) - \frac{\delta p}{\rho} g + \frac{ik_x B_0}{4\pi \rho} \delta B_z,$$

$$\frac{\delta p}{\rho} = \gamma \frac{\delta p}{\rho} - \xi_z \frac{d \ln \rho}{dz} \rho^{-\gamma},$$

but now with magnetic-field perturbations and gradients. The former are given by  $\vec{\delta B} = \vec{\nabla} \times (\vec{\xi} \times \vec{B}_0) \Rightarrow \delta B_x = -\frac{d}{dz} (\xi_z B_0)$

$$\delta B_z = ik_x B_0 \xi_z$$

First, note that  $\frac{\delta p + B_0 \delta B_x / 4\pi}{\rho}$

$$= a^2 \frac{\delta p}{\rho} - \frac{a^2}{\gamma} \frac{d \ln \rho}{dz} \rho^{-\gamma} \xi_z + \frac{B_0}{4\pi \rho} \left( -\frac{d}{dz} \right) (\xi_z B_0)$$

$$= a^2 \left[ -ik_x \xi_x - \xi_z' - \xi_z \frac{d \ln \rho}{dz} - \frac{1}{\gamma} \frac{d \ln \rho}{dz} \rho^{-\gamma} \xi_z - \frac{V_{A0}^2}{a^2} \left( \xi_z' + \xi_z \frac{d \ln B_0}{dz} \right) \right]$$

$$= a^2 \left[ -ik_x \xi_x - \xi_z' \left( 1 + \frac{V_{A0}^2}{a^2} \right) + \frac{g}{a^2} \xi_z \right]$$

Then

$$-\omega^2 \xi_x = -ik_x a^2 \left[ -ik_x \xi_x - \xi_z' \left( 1 + \frac{V_{A0}^2}{a^2} \right) + \frac{g}{a^2} \xi_z \right] + \frac{ik_x B_0}{4\pi \rho} \left[ -\xi_z' B_0 - \xi_z B_0 \frac{d \ln B_0}{dz} \right] + \frac{ik_x B_0}{4\pi \rho} \xi_z \frac{dB_0}{dz}$$

$$\Rightarrow (-\omega^2 + k_x^2 a^2) \xi_x = ik_x a^2 \xi_z' \left( 1 + \frac{V_{A0}^2}{a^2} \right) - ik_x g \xi_z - \cancel{ik_x V_{A0}^2 \xi_z'} - \cancel{ik_x V_{A0}^2 \frac{d \ln B_0}{dz} \xi_z} + \cancel{ik_x V_{A0}^2 \frac{d \ln B_0}{dz} \xi_z}$$

$$\Rightarrow \boxed{(-\omega^2 + k_x^2 a^2) \xi_x = ik_x a^2 \xi_z' - ik_x \xi_z g} \quad \text{Same as } \textcircled{\#} \text{ w/o B field!}$$

$$\Rightarrow \frac{\delta p}{\rho} = -\frac{\xi_z'}{\xi_z} - \xi_z \frac{d \ln \rho}{dz} - \frac{ik_x \left[ ik_x a^2 \xi_z' - ik_x \xi_z g \right]}{(-\omega^2 + k_x^2 a^2)}$$



$$\frac{\delta p}{\rho} = \frac{\omega^2 \xi_z' + \left[ (\omega^2 - k_x^2 a^2) \frac{d\mu p}{dz} - k_x^2 g \right] \xi_z}{k_x^2 a^2 - \omega^2} \quad \left. \begin{array}{l} \text{same as } \odot \\ \text{w/o B field!} \end{array} \right\}$$

Buoyancy is fundamentally the same as in the hydro case. Plugging all of this into the z-component of the momentum eqn. gives (with  $\frac{\delta p + \delta B^2/8\pi}{\rho} = \frac{\omega^2}{\omega^2 - k_x^2 a^2} \left[ \frac{\gamma g \xi_z}{a^2} - \gamma \xi_z' \right] - \frac{\gamma v_{A0}^2 \xi_z'}{a^2}$ )

$$-\omega^2 \xi_z = -\frac{1}{\rho} \frac{d}{dz} \left[ \frac{\rho \omega^2}{\omega^2 - k_x^2 a^2} \left( \frac{\gamma g \xi_z}{a^2} - \gamma \xi_z' \right) - \frac{B_0^2}{4\pi} \xi_z' \right]$$

$$-g \left\{ \frac{\omega^2 \xi_z' + \left[ (\omega^2 - k_x^2 a^2) \frac{d\mu p}{dz} - k_x^2 g \right] \xi_z}{k_x^2 a^2 - \omega^2} \right\}$$

$$+ \frac{ik_x B_0}{4\pi \rho} (ik_x B_0 \xi_z)$$

$$\Rightarrow (-\omega^2 + k_x^2 v_{A0}^2) \xi_z = -\frac{a^2}{\gamma} \frac{d\mu p}{dz} \frac{\omega^2}{\omega^2 - k_x^2 a^2} \left( \frac{\gamma g \xi_z}{a^2} - \gamma \xi_z' \right)$$

$$- \frac{a^2}{\gamma} \frac{\omega^2 k_x^2 a^2}{(\omega^2 - k_x^2 a^2)^2} \frac{d\mu T}{dz} \left( \frac{\gamma g \xi_z}{a^2} - \gamma \xi_z' \right) + v_{A0}^2 \xi_z'' + \xi_z' v_{A0}^2 \frac{d\ln B_0^2}{dz}$$

$$- \frac{a^2}{\gamma} \frac{\omega^2}{\omega^2 - k_x^2 a^2} \left( \frac{\gamma g \xi_z'}{a^2} - \frac{\gamma g \xi_z}{a^2} \frac{d\ln T}{dz} - \gamma \xi_z'' \right)$$

$$+ \frac{g}{\omega^2 - k_x^2 a^2} \left\{ \frac{\omega^2 \xi_z'}{a^2} + \left[ (\omega^2 - k_x^2 a^2) \frac{d\mu p}{dz} - k_x^2 g \right] \xi_z \right\}$$

After some straightforward algebra, we find

$$\begin{aligned} & \frac{6}{\epsilon_2} \left( V_{A0}^2 + \frac{a^2 \omega^2}{\omega^2 - k_x^2 a^2} \right) \\ & + \frac{6}{\epsilon_2} \left[ V_{A0}^2 \frac{d \ln B_0^2}{dz} + \frac{a^2 \omega^2}{(\omega^2 - k_x^2 a^2)^2} \left( \omega^2 \frac{d \ln \rho}{dz} - k_x^2 a^2 \frac{d \ln \rho}{dz} \right) \right] \\ & + \frac{6}{\epsilon_2} \left[ \frac{g}{\gamma} \frac{k_x^2 a^2}{\omega^2 - k_x^2 a^2} \left( \frac{d \ln \rho \rho^{-\gamma}}{dz} + \frac{\gamma V_{A0}^2}{a^2} \frac{d \ln B_0}{dz} \right) - \frac{g \omega^2 k_x^2 a^2}{(\omega^2 - k_x^2 a^2)^2} \frac{d \ln T}{dz} \right] \\ & + \omega^2 - k_x^2 V_{A0}^2 \end{aligned}$$

All extra terms  $\propto V_{A0}^2$ . For  $k_z \rightarrow 0$ ,  $k_x^2 a^2 \gg 1$ , this becomes

$$\omega^2 \approx \underbrace{k_x^2 V_{A0}^2}_{\text{magnetic tension}} + \frac{g}{\gamma} \left( \underbrace{\frac{d \ln \rho \rho^{-\gamma}}{dz}}_{\text{thermal buoyancy}} + \frac{1}{\beta} \underbrace{\frac{d \ln B_0^2}{dz}}_{\text{magnetic buoyancy}} \right) \quad \text{w/ } \beta = \frac{B_0^2}{8\pi p}$$

Now,  $\frac{d \ln \rho \rho^{-\gamma}}{dz} + \frac{1}{\beta} \frac{d \ln B_0^2}{dz} > 0$  for stability. But the physics is the same as in Schwarzschild convection — just the pressure balance is different.

## • Rotational and magnetorotational instability.

Accretion disks are ubiquitous in astrophysics, and they get their namesake by actually facilitating mass accretion onto compact objects like young protostars, neutron stars, black holes, etc. For this to happen, angular momentum must be redistributed, and it turns out that this is frustratingly difficult in Keplerian disks. The problem is that, hydrodynamically, Keplerian flows are quite stable (we'll show below that they are linearly stable; there is no proof that they are nonlinearly stable, but experimental efforts to find nonlinear instability in hydrodynamic, differentially rotating flows have so far failed). Fluid elements do not like to give up their angular momentum. The culprit is the Coriolis force, a surprisingly strong stabilizing effect. (Indeed, planar shear flows without rotation quite easily disrupt so long as the viscosity is not too large.) Another issue is that the molecular viscosity, which might transport angular momentum purely by frictional means, is absolutely negligible in most all astrophysical fluids. One way out is to posit some anomalous viscosity via (unknown) turbulence. This is the route taken in the classic Shakura & Sunyaev

(1973) paper — assume turbulent transport, characterize it by a scalar viscosity, and take that viscous stress to be proportional to the gas pressure:

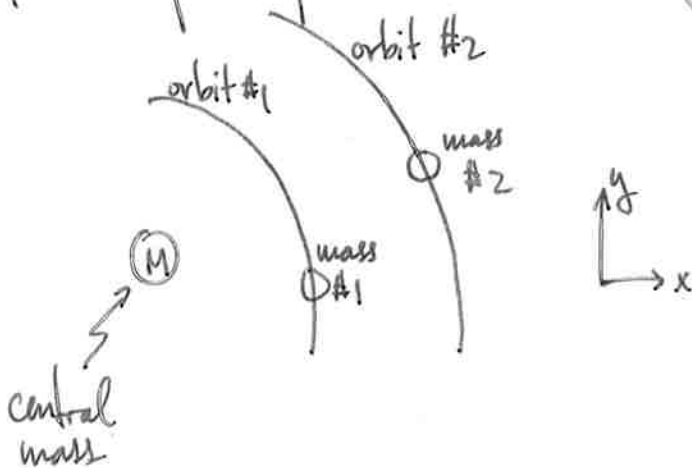
$$T_{r\phi} = \alpha_{ss} P$$

$\swarrow$  gas pressure  
 $\nwarrow$  proportionality constant

$r$ - $\phi$  component of the stress tensor, responsible for transporting  $\phi$  momentum in the  $r$  direction.

This led to the " $\alpha$ -disk" framework of accretion disks, which has been extremely profitable, but woefully unsatisfying. This changed in 1991.

Let's pause here and explore the above claims a bit further. I said a Keplerian disk is hydrodynamically stable to small disturbances. Let's prove it. There are two ways to do this — using point masses in orbits, and using the full hydro eqs. in a rotating frame. Here's the first:



The eqns. of motion for these masses are

$$\ddot{x} - 2\Omega \dot{y} = -\frac{d\Omega^2}{d\ln R} x$$

$$\ddot{y} + 2\Omega \dot{x} = 0$$

These are called the "Hill equations" (Hill 1878). They include the Coriolis force and an extra term in the "radial" equation for the  $x$  displacement that accounts for the "tidal" force (the <sup>local</sup> difference between the centrifugal force and gravity). And they are local — note the Cartesian coordinate system with  $x$  pointing locally radial and  $y$  pointing locally azimuthal. Take solutions  $x, y \sim e^{i\omega t}$  to compute the normal modes of this system:

$$\begin{bmatrix} -\omega^2 + \frac{d\Omega^2}{d\ln R} & 2\Omega i\omega \\ -2\Omega i\omega & -\omega^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \omega^2 \left( \omega^2 - \frac{d\Omega^2}{d\ln R} \right) = 4\Omega^2 \omega^2$$

$$\Rightarrow \omega^2 - \underbrace{\left( 4\Omega^2 + \frac{d\Omega^2}{d\ln R} \right)}_{\equiv k^2} = 0 \Rightarrow \boxed{\omega = \pm k}$$

"epicyclic frequency"

These are epicyclic oscillations when  $k^2 > 0$ , and exponentially growing disturbances when  $k^2 < 0$ .

Note that  $4\Omega^2 + \frac{d\Omega^2}{d\ln R} = \frac{1}{R^3} \frac{dl^2}{dR}$ , where  $l = \Omega R^2$  is the (specific) angular momentum. Thus,

$$\frac{dl^2}{d\ln R} > 0 \Leftrightarrow \text{linear stability}$$

"Rayleigh criterion"



The fluid way: let's assume incompressibility for simplicity. Going back to our hydrodynamic eqs., with gravity from a central point mass and  $\vec{u} = \vec{v} + R\Omega\hat{\phi}$ , we have

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) v_R + \vec{v} \cdot \vec{\nabla} v_R - 2\Omega v_\phi - R\Omega^2 - \frac{v_\phi^2}{R} \\ = -\frac{\partial p}{\partial R} \frac{1}{\rho} + g_R, \end{aligned}$$

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) v_\phi + \vec{v} \cdot \vec{\nabla} v_\phi + 2\Omega v_R + v_R \frac{d\Omega}{dlnR} + \frac{v_R v_\phi}{R} \\ = -\frac{1}{R} \frac{\partial p}{\partial \phi} \frac{1}{\rho}, \end{aligned}$$

where  $g_R = -\frac{GM}{R^2}$ . Our equilibrium state is  $\vec{v} = 0$ ,

$p = \text{constant}$ , and  $g_R = -R\Omega^2 \Rightarrow \Omega^2 = \frac{GM}{R^3}$ , a Keplerian orbit. Writing  $\vec{v} = 0 + \delta\vec{v}$  and  $p = p_0 + \delta p$ , our equations to linear order in  $\delta$  are

$$\left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) \delta v_R - 2\Omega \delta v_\phi = -\frac{1}{\rho} \frac{\partial}{\partial R} \delta p,$$

$$\left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) \delta v_\phi + 2\Omega \delta v_R + \frac{d\Omega}{dlnR} \delta v_R = -\frac{1}{\rho} \frac{1}{R} \frac{\partial \delta p}{\partial \phi}.$$

For simplicity, let us neglect the  $\partial/\partial t$  derivatives and let  $\delta v \sim \exp(-i\omega t + ik_R R + ik_z z)$ . Then, with

$$\nabla \cdot \delta \mathbf{v} = 0 \Rightarrow k_R \delta v_R = -k_z \delta v_z \quad \text{and} \quad \frac{\partial}{\partial t} \delta v_z = -\frac{1}{\rho} \frac{\partial}{\partial z} \delta p,$$

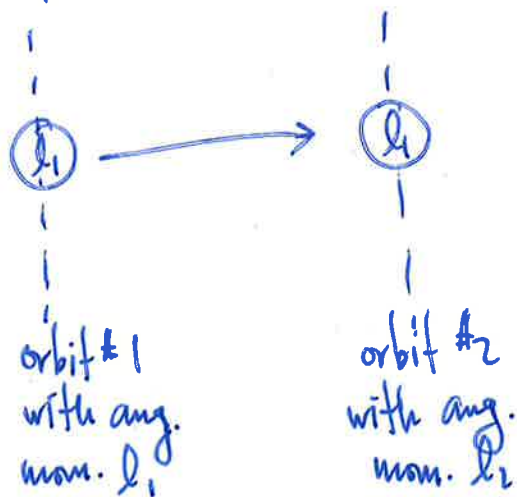
we find  $\frac{\delta p}{\rho} = -\frac{k_R}{k_z^2} \omega \delta v_R$  and so

$$\begin{bmatrix} -i\omega \frac{k^2}{k_z^2} & -2\Omega \\ \frac{k^2}{2\Omega} & -i\omega \end{bmatrix} \begin{bmatrix} \delta v_R \\ \delta v_z \end{bmatrix} = 0 \Rightarrow \boxed{\omega^2 = \frac{k_z^2}{k^2} \kappa^2}$$

$\uparrow = 2\Omega + \frac{d\Omega}{d\ln R}$

Same stability criterion as before,  $\kappa^2 > 0$ .

Physically, what's going on?



Take a fluid element at orbit #1 and displace it outwards to orbit #2 while maintaining a constant ang. momentum. Since  $l_2 > l_1$ , the fluid element cannot stay in its new orbit and must return back to orbit #1.  $\Rightarrow$  STABLE.

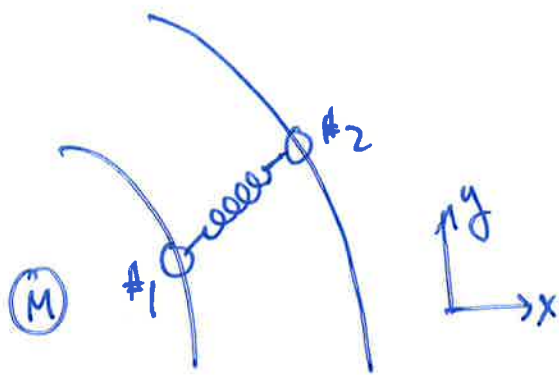
$\rightarrow R$  (outwards in disk)

Now, back to 1991 ...



Steve Balbus and John Hawley, then both at Univ. of Virginia, found by a straightforward linear analysis and clever use of 90's supercomputers, that a small but finite magnetic field is all that is required to linearly destabilize Keplerian rotation. How could this be missed? The answer is complicated. The instability — at first known as the "Balbus-Hawley instability" but now goes by the moniker "magnetorotational instability" (MRI) — appeared in a little-known Russian paper by Velikhov in 1959, and 2 years later made its way into Chandrasekhar's classic text on "hydrodynamic and hydromagnetic stability". But there it appeared in a rather odd guise, at least to an astronomer thinking about accretion disks — Couette flow, i.e., rotational flow excited by placing a (conducting) fluid between two cylindrical walls rotating at different speeds. It wasn't until B&H rediscovered it and placed it in the astrophysical context that the instability became appreciated as a possible solution to the accretion problem. What followed was an industry of linear analysis and nonlinear numerical simulations aiming to characterize the MRI in a wide variety of disk systems. But let's go back to the beginning:





Take the Hill system but attach a spring between the two masses — magnetic fields act as springs, so you can imagine this being a field line threading two fluid elements. Add Hooke's law to the eqns. of motion:

$$\ddot{x} - 2\Omega y = -\frac{d\Omega^2}{d\ln R} x - Kx$$

$$\ddot{y} + 2\Omega x = -Ky$$

w/  $K = \text{spring constant}$

$$x, y \sim e^{i\omega t} \Rightarrow \begin{bmatrix} -\omega^2 + \frac{d\Omega^2}{d\ln R} + K & 2\Omega i\omega \\ -2\Omega i\omega & -\omega^2 + K \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \left( \omega^2 - \frac{d\Omega^2}{d\ln R} - K \right) (\omega^2 - K) = 4\Omega^2 \omega^2$$

$$\Rightarrow (\omega^2 - K) \left( \omega^2 - K - 4\Omega^2 - \frac{d\Omega^2}{d\ln R} \right) = 4\Omega^2 K$$

↑ extra    
 ↑ extra    
 ↑ extra

Solutions are  $\omega^2 - K = \frac{K^2}{2} \pm \sqrt{\left(\frac{K^2}{2}\right)^2 + 4\Omega^2 K}$

whose (-) solution is unstable if

$$K + \frac{d\Omega^2}{d\ln R} < 0$$

This is important, because Keplerian disks have  $\frac{d\Omega^2}{dlnR} < 0$ ! Note that the spring cannot be too strong here. Interestingly,

$$\boxed{\frac{d\Omega^2}{dlnR} > 0 \text{ for hydro stability} \rightarrow \frac{d\Omega^2}{dlnR} > 0 \text{ for MHD stability}}$$

One can show that the Lagrangian change in the ang. mom. of a fluid element as it is displaced is given by

$$\frac{\delta \ell}{\ell} = \frac{x}{R} \left( \frac{K^2}{2\Omega^2} - \frac{i\omega}{\Omega} \frac{y}{x} \right)$$

$$= -\frac{x}{R} \left( \frac{2K}{\omega^2 - K} \right) \left\{ \begin{array}{l} \text{the spring broke} \\ \text{conservation of angular} \\ \text{momentum!} \end{array} \right.$$

If  $K \ll \omega^2 \sim \Omega^2$ , then outward displacements ( $x > 0$ ) gain angular momentum as they are torqued by the spring (NB:  $\omega^2 < 0$  corresponds to growth, so  $\delta \ell \propto (x/R)$ ).

At max. growth (take  $\frac{\partial}{\partial K}$  of disp. relation and find extrema of  $\omega^2$ ), the growth rate  $-i\omega = \frac{1}{2} \left| \frac{d\Omega}{dlnR} \right|$  and

$$\frac{\delta l}{l} \Big|_{\max} = \frac{2x}{R} \left( 1 - \frac{1}{4} \left| \frac{d\Omega}{d \ln R} \right| \right)$$

Now the fluid picture: (let  $\vec{B}_0 = B_0 \hat{z}$  for simplicity)

$$\left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) \delta v_R - 2\Omega \delta v_\varphi = -\frac{1}{\rho} \frac{\partial}{\partial R} \left( \delta p + \frac{\vec{B}_0 \cdot \delta \vec{B}}{4\pi} \right) + \frac{\vec{B}_0 \cdot \vec{\nabla} \delta B_R}{4\pi\rho}$$

$$\left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) \delta v_\varphi + \frac{k^2}{2\Omega} \delta v_R = -\frac{1}{\rho} \frac{1}{R} \frac{\partial}{\partial \varphi} \left( \delta p + \frac{\vec{B}_0 \cdot \delta \vec{B}}{4\pi} \right) + \frac{\vec{B}_0 \cdot \vec{\nabla} \delta B_\varphi}{4\pi\rho}$$

We need eqns. for  $\delta B_R$  and  $\delta B_\varphi$  ( $\delta B_z$  is determined from  $\vec{\nabla} \cdot \delta \vec{B} = 0$ ). So, take the induction eqn. and write it in cylindrical coordinates w/  $\vec{u} = \vec{v} + R\Omega \hat{\varphi}$ :

$$\left( \vec{B} \cdot \vec{\nabla} \vec{u} = \hat{e}_i \vec{B} \cdot \vec{\nabla} v_i + \frac{v_R B_\varphi}{R} \hat{\varphi} - \frac{v_\varphi B_R}{R} \hat{r} - \Omega B_\varphi \hat{r} + \hat{\varphi} B_R \left( \Omega + \frac{d\Omega}{d \ln R} \right) \right)$$

$$\Rightarrow \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) \delta B_R = \vec{B}_0 \cdot \vec{\nabla} \delta v_R,$$

$$\left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) \delta B_\varphi = \vec{B}_0 \cdot \vec{\nabla} \delta v_\varphi + \delta B_R \frac{d\Omega}{d \ln R}.$$

Again, take  $\partial/\partial\varphi = 0$  and let  $\delta \sim \exp(-i\omega t + ik_r R + ik_z z)$ .

$$\left. \begin{aligned} \vec{\nabla} \cdot \vec{\delta v} = 0 &\rightarrow \delta v_z = -\frac{k_r}{k_z} \delta v_r \\ \vec{\nabla} \cdot \vec{\delta B} = 0 &\rightarrow \delta B_z = -\frac{k_r}{k_z} \delta B_r \end{aligned} \right\} \begin{aligned} \frac{\partial}{\partial t} \delta v_z &= -\frac{1}{\rho} \frac{\partial}{\partial z} \left( \delta p + \frac{\vec{B}_0 \cdot \vec{\delta B}}{4\pi} \right) \\ &\quad + \frac{\vec{B}_0 \cdot \vec{\nabla}}{4\pi\rho} \delta B_z \end{aligned}$$

$$\rightarrow \frac{\delta p + \vec{B}_0 \cdot \vec{\delta B}}{\rho} = -\frac{\omega k_r}{k_z^2} \delta v_r - \frac{\vec{k} \cdot \vec{B}_0}{4\pi\rho} \frac{k_r}{k_z^2} \delta B_r$$

$$\Rightarrow \left\{ \begin{aligned} -i\omega \delta v_r - 2\Omega \delta v_\varphi &= -ik_r \left[ -\frac{\omega k_r}{k_z^2} \delta v_r - \frac{\vec{k} \cdot \vec{B}_0}{4\pi\rho} \frac{k_r}{k_z^2} \delta B_r \right] \\ &\quad + \frac{i\vec{k} \cdot \vec{B}_0}{4\pi\rho} \delta B_r \\ -i\omega \delta v_\varphi + \frac{k^2}{2\Omega} \delta v_r &= \frac{i\vec{k} \cdot \vec{B}_0}{4\pi\rho} \delta B_\varphi \end{aligned} \right.$$

$$\Rightarrow \left\{ \begin{aligned} -i\omega \frac{k^2}{k_z^2} \delta v_r - 2\Omega \delta v_\varphi &= \frac{i\vec{k} \cdot \vec{B}_0}{4\pi\rho} \frac{k^2}{k_z^2} \delta B_r \\ \frac{k^2}{2\Omega} \delta v_r - i\omega \delta v_\varphi &= \frac{i\vec{k} \cdot \vec{B}_0}{4\pi\rho} \delta B_\varphi \end{aligned} \right.$$

and  $\left\{ \begin{aligned} -i\omega \delta B_r &= i\vec{k} \cdot \vec{B}_0 \delta v_r \\ -i\omega \delta B_\varphi &= i\vec{k} \cdot \vec{B}_0 \delta v_\varphi + \delta B_r \frac{d\Omega}{d\ln R} \end{aligned} \right\}$  solve these for  $\vec{\delta v}$  and plug into these



$$\begin{cases} -i\omega \frac{k^2}{k^2} \left( \frac{-i\omega \delta B_R}{ik \cdot B_0} \right) - 2\Omega \left( \frac{-i\omega \delta B_y - \delta B_R \frac{d\Omega}{d\ln R}}{ik \cdot B_0} \right) = \frac{ik \cdot B_0}{4\pi\mu} \frac{k^2}{k^2} \delta B_R \\ \frac{k^2}{2\Omega} \left( \frac{-i\omega \delta B_R}{ik \cdot B_0} \right) - i\omega \left( \frac{-i\omega \delta B_y - \delta B_R \frac{d\Omega}{d\ln R}}{ik \cdot B_0} \right) = \frac{ik \cdot B_0}{4\pi\mu} \delta B_y \end{cases}$$

Cleaning up...

$$\begin{bmatrix} -\omega^2 + (k \cdot v_A)^2 - \frac{k^2}{k^2} \frac{d\Omega^2}{d\ln R} & 2\Omega i\omega \frac{k^2}{k^2} \\ -2\Omega i\omega & -\omega^2 + (k \cdot v_A)^2 \end{bmatrix} \begin{bmatrix} \delta B_R \\ \delta B_y \end{bmatrix} = 0.$$

Look familiar?  $K \rightarrow (k \cdot v_A)^2$ ! Magnetic tension is a spring. Dispersion relation:

$$\begin{aligned} [\omega^2 - (k \cdot v_A)^2] \left[ \omega^2 - (k \cdot v_A)^2 - k^2 \frac{k^2}{k^2} \right] &= 4\Omega^2 (k \cdot v_A)^2 \\ \Rightarrow \omega^2 - (k \cdot v_A)^2 &= \frac{k^2}{2} \pm \sqrt{\left(\frac{k^2}{2}\right)^2 + 4\Omega^2 (k \cdot v_A)^2} \end{aligned}$$

unstable if  $\boxed{(k \cdot v_A)^2 + \frac{d\Omega^2}{d\ln R} < 0}$

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Note: Can write discriminant as  $\left[ \frac{k^2}{2} + (k \cdot v_A)^2 \right]^2 - (k \cdot v_A)^2 \left[ (k \cdot v_A)^2 + k^2 - 4\Omega^2 \right]$   
 $= \left[ \frac{k^2}{2} + (k \cdot v_A)^2 \right]^2 - (k \cdot v_A)^2 \left[ (k \cdot v_A)^2 + \frac{d\Omega^2}{d\ln R} \right]$

This means Keplerian disks are <sup>linearly</sup> unstable, provided the magnetic field isn't so strong that all the wavenumbers  $k_z = 2\pi/\lambda_z$  that can fit within the height of the disk satisfy  $k_z^2 v_A^2 > \left| \frac{d\Omega^2}{d\ln R} \right|$  — then tension stabilizes all relevant modes.

See Balbus & Hawley 1998 Rev. Mod. Phys. for more.

## PART V

# Single-particle motion

So far, we have concerned ourselves with the response of fluid elements to both imposed and self-consistently generated electromagnetic and gravitational fields. But those fluid elements are composed of charged (and neutral) particles; it would be good to know how those particles move through phase space. Now, we all know Newton's equations of motion for a particle in the presence of electric and magnetic fields:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = \frac{q}{m} \left[ \mathbf{E}(t, \mathbf{r}) + \frac{\mathbf{v}}{c} \times \mathbf{B}(t, \mathbf{r}) \right]. \quad (\text{V.1})$$

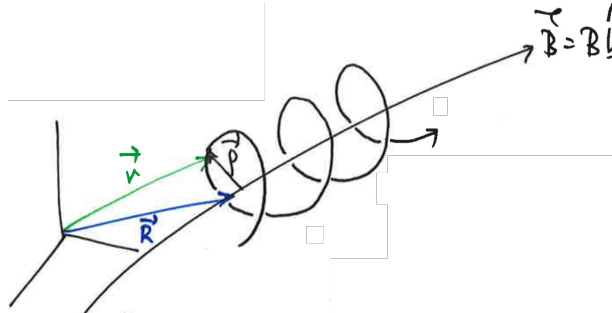
But solutions to (V.1) are surprisingly subtle, even in seemingly simple situations. . .

### V.1. Particle motion in uniform electric and magnetic fields

Consider the motion of a single, charged particle. Start by decomposing the particle's position into a Larmor position  $\boldsymbol{\rho}$  and a guiding-center position  $\mathbf{R}$ , *viz.*,

$$\mathbf{r} = \boldsymbol{\rho} + \mathbf{R} = -\frac{\mathbf{v} \times \hat{\mathbf{b}}}{\Omega} + \mathbf{R}, \quad (\text{V.2})$$

where  $\Omega \doteq qB/mc$  is the Larmor frequency:



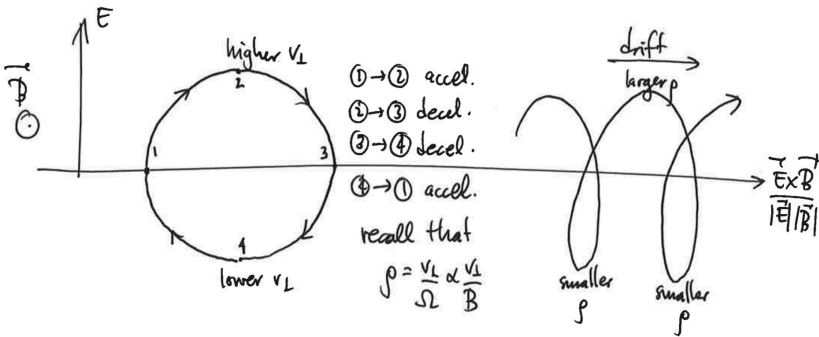
The Larmor position just oscillates around the guiding center at a rate  $\dot{\vartheta} \simeq -\Omega$  (more on this later). Using this decomposition, let's begin with something relatively simple: particle motion in constant electric and magnetic fields.

Rearranging (V.2) and taking the time derivative,

$$\begin{aligned} \dot{\mathbf{R}} &= \dot{\mathbf{r}} - \dot{\boldsymbol{\rho}} \\ &= \mathbf{v} + \frac{d\mathbf{v}}{dt} \times \frac{\hat{\mathbf{b}}}{\Omega} \\ &= \mathbf{v} + \frac{q}{m} \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \times \frac{\hat{\mathbf{b}}}{\Omega} \quad (\text{using (V.1)}) \\ &= \mathbf{v} + \frac{q\mathbf{E} \times \hat{\mathbf{b}}}{m\Omega} - \mathbf{v}_{\perp} \\ &= v_{\parallel} \hat{\mathbf{b}} + \frac{c\mathbf{E} \times \mathbf{B}}{B^2} \doteq v_{\parallel} \hat{\mathbf{b}} + \mathbf{v}_E \\ &= \text{parallel streaming of the guiding center} + \text{'E cross B drift'} \end{aligned} \quad (\text{V.3})$$

Note that the perpendicular drift is charge independent; ions and electrons drift in the same direction with the same speed. Thus, no currents are generated by this type of

guiding-center drift. The physical origin of the  $\mathbf{E} \times \mathbf{B}$  drift is the dependence of the gyroradius of a particle on  $v_{\perp}$ , which periodically changes due to acceleration by the perpendicular component of the electric field:



You'll see when we study ideal MHD that particles  $\mathbf{E} \times \mathbf{B}$  drift in order to stay on a given magnetic-field line.

For more a general force  $\mathbf{F}$ , the perpendicular drift is

$$\mathbf{v}_F \doteq \frac{\mathbf{F} \times \hat{\mathbf{b}}}{m\Omega}, \tag{V.4}$$

which is generally charge dependent and thus results in currents.

### V.2. Particle motion in a non-uniform magnetic field

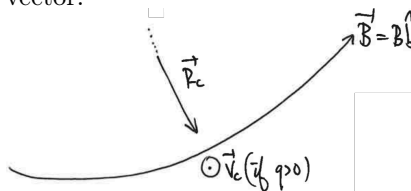
Next, let's keep the uniform electric field, but allow the magnetic field to vary in space. Equation (V.3) acquires an additional term due to gradients in the magnetic field along the particle orbit:

$$\dot{\mathbf{R}} = v_{\parallel} \hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v} \times \frac{d \hat{\mathbf{b}}}{dt \Omega}. \tag{V.5}$$

The final term in (V.5) includes two new drifts, which can be obtained rigorously using 'guiding-center theory' (and we will, in §V.4). But they can also be obtained quite readily if you already know their names: 'curvature drift' and 'grad-B drift'. The former suggests we look at the centrifugal force on a particle as it follows a curved magnetic-field line:

$$\mathbf{F}_c = \frac{mv_{\parallel}^2}{r_c} \hat{\mathbf{r}}_c, \quad \text{where } \hat{\mathbf{r}}_c = -r_c \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \tag{V.6}$$

with  $r_c$  being the radius of curvature of the field line. The unit vector  $\hat{\mathbf{r}}_c$  points in the direction of the curvature vector:



Feeding (V.6) into (V.4), we obtain the curvature drift,

$$\mathbf{v}_c \doteq \frac{\mathbf{F}_c \times \hat{\mathbf{b}}}{m\Omega} = -\frac{v_{\parallel}^2}{\Omega} (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \times \hat{\mathbf{b}}. \tag{V.7}$$

Note that it is charge dependent.



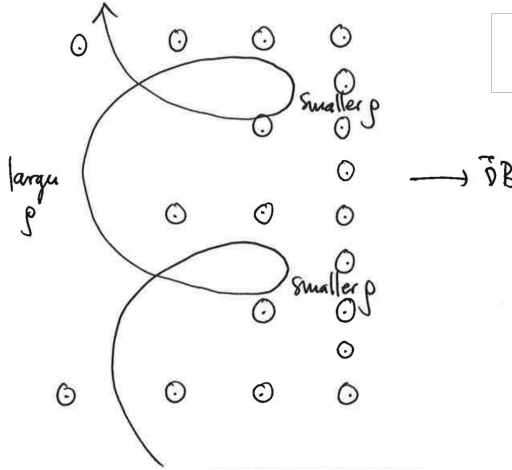
As for the ‘grad- $B$  drift’, imagine a magnetic dipole with moment

$$\boldsymbol{\mu} = \frac{1}{2} q \mathbf{r} \times \frac{\mathbf{v}}{c} = -\frac{1}{2} q \rho \frac{v_{\perp}}{c} \hat{\mathbf{b}} = -\frac{m v_{\perp}^2}{2B} \hat{\mathbf{b}}, \quad (\text{V.8})$$

exposed to an inhomogeneous magnetic field. The force on a dipole is equal to  $\nabla(\boldsymbol{\mu} \cdot \mathbf{B}) = -\boldsymbol{\mu} \nabla B$ , and so (using (V.4)), there is a drift given by

$$\mathbf{v}_{\nabla B} \doteq \frac{-\boldsymbol{\mu} \nabla B \times \hat{\mathbf{b}}}{m \Omega} = \frac{v_{\perp}^2}{2\Omega} \hat{\mathbf{b}} \times \nabla \ln B. \quad (\text{V.9})$$

This drift results from the increase (decrease) in the gyroradius of a particle as the particle enters a region of decreased (increased) magnetic-field strength:



The grad- $B$  drift is also charge dependent.

Note that, in a force-free field configuration with  $\nabla \times \mathbf{B} \parallel \mathbf{B}$ , we have  $\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} = \nabla_{\perp} \ln B$ . Thus, from (V.7) and (V.9),

$$\mathbf{v}_{\text{curv}} + \mathbf{v}_{\nabla B} = \frac{v_{\parallel}^2 + v_{\perp}^2/2}{\Omega} \hat{\mathbf{b}} \times \nabla \ln B.$$

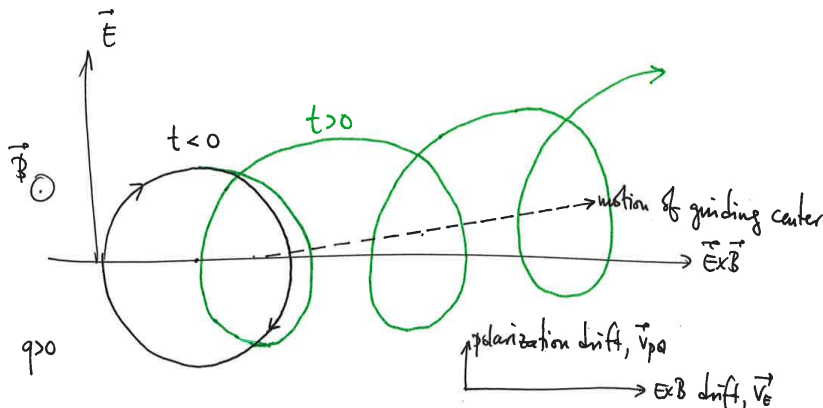
Averaged over all particles, these drifts are  $\sim v_{\text{th}}(\rho/\ell_B)$ , which is typically (very!) sub-thermal.

### V.3. Particle motion in a time-dependent electric field

If  $\mathbf{E}$  has some explicit time dependence, then there is yet another drift called ‘polarization drift’, which can be thought of as being due to an inertial force  $-m d\mathbf{v}_E/dt$  on the guiding center:

$$\mathbf{v}_{\text{pol}} \doteq -\frac{d\mathbf{v}_E}{dt} \times \frac{\hat{\mathbf{b}}}{\Omega} = \frac{1}{\Omega} \frac{c}{B} \frac{\partial \mathbf{E}_{\perp}}{\partial t}. \quad (\text{V.10})$$

If an electric field is suddenly switched on in a plasma, the ions will drift faster than the electrons (!), thus polarizing the plasma. The idea here is that, if the electric field varies as the particle navigates its gyro-orbit and does not average to zero, the result is a net shift of the guiding center in the direction of  $\partial \mathbf{E}_{\perp}/\partial t$  for positive charges and in the opposite direction for negative charges. The simplest way to picture this is to consider switching on a linearly increasing perpendicular electric field at  $t = 0$ :



Because the ions and electrons have different signs of polarization drift, there is a current produced:

$$\mathbf{j}_{\text{pol}} = \rho \left( \frac{c}{B} \right)^2 \frac{\partial \mathbf{E}_{\perp}}{\partial t}, \quad (\text{V.11})$$

where  $\rho \doteq m_i n_i + m_e n_e$  is the mass density. This current is dominated by the heavier species (ions), since that species has a larger gyro-period and thus is displaced by a much larger distance by the changing electric field during each orbit. By analogy with standard electrodynamics in dielectric media, in which<sup>6</sup>

$$\mathbf{j}_{\text{pol}} = \frac{\varepsilon}{4\pi} \frac{\partial \mathbf{E}}{\partial t},$$

we see that the effective permittivity  $\varepsilon = (c/v_A)^2$ , where  $v_A \doteq B/\sqrt{4\pi\rho}$  is the Alfvén speed. (Polarization current is tied to the propagation of Alfvén waves.) Since we often have  $c/v_A \gg 1$ , most plasmas have  $\varepsilon \gg 1$ , i.e., they behave as strongly polarizable media.

#### V.4. Guiding-center theory

*This is optional material. It is a beautiful exercise in asymptotic expansion, but there isn't enough time in this course for go through the procedure in detail. I encourage to you read it on your own and work through the calculations. You'll be better for it.*

There is a systematic way of deriving drifts that are due to the non-constantly of forces along a particle's orbit, so long as these forces vary slowly. By that, we mean that the length scales ( $\ell$ ) and time scales ( $\tau$ ) over which the forces vary are long compared to  $\rho$  and  $\Omega^{-1}$ , respectively:

$$\frac{\rho}{\ell} \ll 1, \quad (\Omega\tau)^{-1} \ll 1.$$

To enact this scale hierarchy, we introduce a small parameter,

$$\epsilon \doteq \frac{\rho}{\ell} \sim (\Omega\tau)^{-1},$$

and expand (V.1) in powers of  $\epsilon$ . Not surprisingly, we will find a fast gyromotion and a slow guiding-center motion.

<sup>6</sup>Because of the standard undergraduate training in electromagnetism, you may not be familiar with dielectrics in Gaussian units. If that's true, then note the following conventions:  $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \doteq \varepsilon\mathbf{E} \doteq (1 + 4\pi\chi_e)\mathbf{E}$ .

Start by writing  $\mathbf{R} \doteq \mathbf{r} - \boldsymbol{\rho}$  as before, but now with the Larmor vector defined by

$$\boldsymbol{\rho} = -\frac{(\mathbf{v} - \mathbf{v}_E) \times \hat{\mathbf{b}}}{\Omega}. \quad (\text{V.12})$$

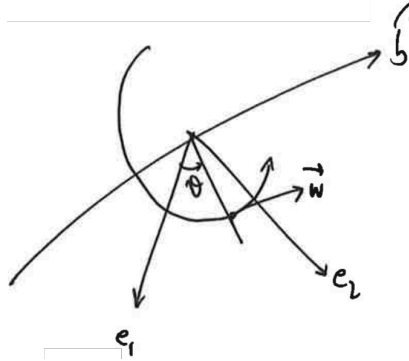
The reason for separating out  $\mathbf{v}_E$  from the other drifts is that the  $\mathbf{E} \times \mathbf{B}$  is *not* small in  $\epsilon$ . (Indeed, this is why  $\mathbf{E} \times \mathbf{B}$  motion plays such a prominent role in MHD.) For ease of notation, write

$$\mathbf{w} \doteq \mathbf{v} - \mathbf{v}_E, \quad (\text{V.13})$$

so that  $\boldsymbol{\rho} = -\mathbf{w} \times \hat{\mathbf{b}}/\Omega$ . Now, we know that the directions parallel ( $\parallel$ ) and perpendicular ( $\perp$ ) to the magnetic field behave differently (certainly in the  $\epsilon \ll 1$  limit), so write

$$\mathbf{w} = v_{\parallel} \hat{\mathbf{b}} + \mathbf{w}_{\perp} = v_{\parallel} \hat{\mathbf{b}} + w_{\perp} (\hat{\mathbf{e}}_1 \cos \vartheta + \hat{\mathbf{e}}_2 \sin \vartheta), \quad (\text{V.14})$$

where  $\vartheta$  is the gyrophase:



The coordinates  $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{b}})$  are functions of  $(t, \mathbf{r})$  as the particle sweeps around the changing, inhomogeneous magnetic field. What follows is a gradual shift of the particle coordinates from  $(\mathbf{r}, \mathbf{v})$  to  $(\mathbf{R}, v_{\parallel}, w_{\perp}, \vartheta)$ .

Let us first examine the motion of the guiding-center position, which follows from (V.1) and (V.12):

$$\dot{\mathbf{R}} = \dot{\mathbf{r}} - \dot{\boldsymbol{\rho}} = \underbrace{v_{\parallel} \hat{\mathbf{b}}}_{\textcircled{0}} + \underbrace{\mathbf{v}_E}_{\textcircled{0}} - \underbrace{\frac{d\mathbf{v}_E}{dt} \times \frac{\hat{\mathbf{b}}}{\Omega}}_{\textcircled{1}} + \underbrace{\mathbf{w} \times \frac{d\hat{\mathbf{b}}}{dt}}_{\textcircled{1}}. \quad (\text{V.15})$$

The order in  $\epsilon$  of each term (relative to  $v_{\text{th}}$ ) has been noted. To leading order, there is parallel streaming and the  $\mathbf{E} \times \mathbf{B}$  drift. The next-order terms are those dependent upon spatiotemporal changes in the electromagnetic fields along the particle's trajectory.

Next, the evolution of the parallel velocity:

$$\dot{v}_{\parallel} = \frac{d}{dt}(\mathbf{v} \cdot \hat{\mathbf{b}}) = \underbrace{\frac{q}{m} E_{\parallel}}_{\textcircled{-1}} + \underbrace{(\mathbf{v}_E + \mathbf{w}) \cdot \frac{d\hat{\mathbf{b}}}{dt}}_{\textcircled{0}},$$

where the ordering is given relative to  $v_{\text{th}}/\tau$ . That  $\mathcal{O}(\epsilon^{-1})$  term is a problem; it says that  $E_{\parallel}$  accelerates particles along field lines on the timescale of a Larmor gyration. Since ions and electrons are accelerated in opposite directions, this would lead to a rapid charge

separation, ultimately violating our assumption of slowly varying fields.  $E_{\parallel}$  must be  $\mathcal{O}(\epsilon)$ :

$$\dot{v}_{\parallel} = \frac{d}{dt}(\mathbf{v} \cdot \hat{\mathbf{b}}) = \underbrace{\frac{q}{m} E_{\parallel}}_{\textcircled{0}} + \underbrace{(\mathbf{v}_E + \mathbf{w}) \cdot \frac{d\hat{\mathbf{b}}}{dt}}_{\textcircled{0}}. \quad (\text{V.16})$$

Following similar steps, one can also show that

$$\dot{w}_{\perp} = -\hat{\mathbf{e}}_{\perp} \cdot \left( \underbrace{v_{\parallel} \frac{d\hat{\mathbf{b}}}{dt}}_{\textcircled{0}} + \underbrace{\frac{d\mathbf{v}_E}{dt}}_{\textcircled{0}} \right). \quad (\text{V.17})$$

All these terms have clean physical interpretations. Parallel electric fields accelerate particles along field lines; the plane of the perpendicular drifts tilts as the particles stream along a varying  $\hat{\mathbf{b}}$ ; and parallel motion can become perpendicular motion if  $\hat{\mathbf{b}}$  changes along the orbit.

It's a bit more work to show that

$$\dot{\vartheta} = \underbrace{-\Omega}_{\textcircled{-1}} - \underbrace{\hat{\mathbf{e}}_2 \cdot \frac{d\hat{\mathbf{e}}_1}{dt}}_{\textcircled{0}} - \underbrace{\frac{\mathbf{w}_{\perp} \times \hat{\mathbf{b}}}{w_{\perp}^2} \cdot \left( v_{\parallel} \frac{d\hat{\mathbf{b}}}{dt} + \frac{d\mathbf{v}_E}{dt} \right)}_{\textcircled{0}}, \quad (\text{V.18})$$

and so I'll show you the steps. (It should be obvious that the dominant term is  $-\Omega$ , i.e.,  $\dot{\vartheta} = -\Omega + \mathcal{O}(\epsilon^0) + \dots$ ) Here are those steps:

$$\begin{aligned} \frac{d\mathbf{w}_{\perp}}{dt} &= \frac{d\mathbf{w}_{\perp}}{dt} \hat{\mathbf{e}}_{\perp} + w_{\perp} \left( \frac{d\hat{\mathbf{e}}_1}{dt} \cos \vartheta + \frac{d\hat{\mathbf{e}}_2}{dt} \sin \vartheta \right) + \underbrace{w_{\perp} (-\hat{\mathbf{e}}_1 \sin \vartheta + \hat{\mathbf{e}}_2 \cos \vartheta) \dot{\vartheta}}_{= -\mathbf{w}_{\perp} \times \hat{\mathbf{b}}} \\ &= -\hat{\mathbf{e}}_{\perp} \hat{\mathbf{e}}_{\perp} \cdot \left( v_{\parallel} \frac{d\hat{\mathbf{b}}}{dt} + \frac{d\mathbf{v}_E}{dt} \right) + w_{\perp} \left( \frac{d\hat{\mathbf{e}}_1}{dt} \cos \vartheta + \frac{d\hat{\mathbf{e}}_2}{dt} \sin \vartheta \right) - (\mathbf{w}_{\perp} \times \hat{\mathbf{b}}) \dot{\vartheta} \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad -\frac{\mathbf{w}_{\perp} \times \hat{\mathbf{b}}}{w_{\perp}^2} \cdot \frac{d\mathbf{w}_{\perp}}{dt} &= \frac{(\mathbf{w}_{\perp} \times \hat{\mathbf{b}}) \cdot \hat{\mathbf{e}}_{\perp} \hat{\mathbf{e}}_{\perp}}{w_{\perp}^2} \cdot \left( v_{\parallel} \frac{d\hat{\mathbf{b}}}{dt} + \frac{d\mathbf{v}_E}{dt} \right) \\ &\quad + \underbrace{\left( -\hat{\mathbf{e}}_1 \cdot \frac{d\hat{\mathbf{e}}_2}{dt} \sin^2 \vartheta + \hat{\mathbf{e}}_2 \cdot \frac{d\hat{\mathbf{e}}_1}{dt} \cos^2 \vartheta \right)}_{\text{since } \hat{\mathbf{e}}_1 \cdot d\hat{\mathbf{e}}_1/dt = \hat{\mathbf{e}}_2 \cdot d\hat{\mathbf{e}}_2/dt = 0} + \dot{\vartheta} \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \dot{\vartheta} &= -\frac{\mathbf{w}_{\perp} \times \hat{\mathbf{b}}}{w_{\perp}^2} \cdot \frac{d\mathbf{w}_{\perp}}{dt} - \hat{\mathbf{e}}_2 \cdot \frac{d\hat{\mathbf{e}}_1}{dt} \quad (\text{since } -\hat{\mathbf{e}}_1 \cdot d\hat{\mathbf{e}}_2/dt = \hat{\mathbf{e}}_2 \cdot d\hat{\mathbf{e}}_1/dt) \\ &= -\Omega - \hat{\mathbf{e}}_2 \cdot \frac{d\hat{\mathbf{e}}_1}{dt} - \frac{\mathbf{w}_{\perp} \times \hat{\mathbf{b}}}{w_{\perp}^2} \cdot \left( v_{\parallel} \frac{d\hat{\mathbf{b}}}{dt} + \frac{d\mathbf{v}_E}{dt} \right). \end{aligned}$$

So, we now have the evolution of  $(\mathbf{R}, v_{\parallel}, w_{\perp}, \vartheta)$ , but it's given in terms of  $(\mathbf{r}, \mathbf{v})$ . To proceed, we must write the latter in terms of the former.

To do that, we Taylor expand about the guiding-center position; e.g.,

$$\hat{\mathbf{b}}(t, \mathbf{r}) = \hat{\mathbf{b}}(t, \mathbf{R}) - \frac{\mathbf{w}_\perp \times \hat{\mathbf{b}}}{\Omega} \cdot \nabla \hat{\mathbf{b}}(t, \mathbf{R}) + \dots \quad (\text{V.19})$$

Also,

$$\begin{aligned} \frac{d}{dt} \Big|_{\mathbf{r}, \mathbf{v}} &= \frac{\partial}{\partial t} \Big|_{\mathbf{R}, v_\parallel, w_\perp, \vartheta} + \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}} \Big|_{t, v_\parallel, w_\perp, \vartheta} + \dot{v}_\parallel \frac{\partial}{\partial v_\parallel} \Big|_{t, \mathbf{R}, w_\perp, \vartheta} + \dot{w}_\perp \frac{\partial}{\partial w_\perp} \Big|_{t, \mathbf{R}, v_\parallel, \vartheta} \\ &+ \dot{\vartheta} \frac{\partial}{\partial \vartheta} \Big|_{t, \mathbf{R}, v_\parallel, w_\perp}. \end{aligned} \quad (\text{V.20})$$

Henceforth, I'll be suppressing the argument  $(t, \mathbf{R})$  on  $\hat{\mathbf{b}}$  and  $\mathbf{v}_E$  and the what's-held-fixed labels on the partial derivatives. Using (V.19) and (V.20), we must evaluate our  $(d/dt)(\mathbf{R}, v_\parallel, w_\perp, \vartheta)$  order by order in  $\epsilon$ .

At  $\mathcal{O}(\epsilon^{-1})$ , we have  $\dot{\vartheta} = -\Omega$ , i.e. Larmor gyration. At  $\mathcal{O}(\epsilon^0)$ ,  $\dot{\mathbf{R}} = v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E$ , which is the same as guiding-center motion in constant fields. Next, work on  $\dot{v}_\parallel$  and  $\dot{w}_\perp$ . Begin by noticing that

$$\dot{\vartheta} \frac{\partial}{\partial \vartheta} = -\Omega \frac{\partial}{\partial \vartheta} + \mathcal{O}(\epsilon^0)$$

is the biggest term in  $d/dt$  (see (V.20)). Thus,

$$\frac{d\hat{\mathbf{b}}}{dt} = \left( \frac{\partial}{\partial t} + v_\parallel \hat{\mathbf{b}} \cdot \nabla + \mathbf{v}_E \cdot \nabla \right) \hat{\mathbf{b}} + \Omega \frac{\partial}{\partial \vartheta} \left( \frac{\mathbf{w}_\perp \times \hat{\mathbf{b}}}{\Omega} \right) \cdot \nabla \hat{\mathbf{b}} + \mathcal{O}(\epsilon), \quad (\text{V.21})$$

$$\frac{d\mathbf{v}_E}{dt} = \left( \frac{\partial}{\partial t} + v_\parallel \hat{\mathbf{b}} \cdot \nabla + \mathbf{v}_E \cdot \nabla \right) \mathbf{v}_E + \underbrace{\Omega \frac{\partial}{\partial \vartheta} \left( \frac{\mathbf{w}_\perp \times \hat{\mathbf{b}}}{\Omega} \right) \cdot \nabla \mathbf{v}_E}_{= \mathbf{w}_\perp} + \mathcal{O}(\epsilon), \quad (\text{V.22})$$

where (to remind you)  $\hat{\mathbf{b}}$  and  $\mathbf{v}_E$  are functions of  $(t, \mathbf{R})$ . (The difference between, say,  $\hat{\mathbf{b}}(t, \mathbf{r})$  and  $\hat{\mathbf{b}}(t, \mathbf{R})$  can be packed into the omitted  $\mathcal{O}(\epsilon)$  terms.) Using (V.21) and (V.22) in the evolution equations (V.16) and (V.17) for  $v_\parallel$  and  $w_\perp$ , respectively, gives

$$\frac{dv_\parallel}{dt} = \frac{qE_\parallel}{m} + (\mathbf{v}_E + \mathbf{w}) \cdot \left( \frac{D\hat{\mathbf{b}}}{Dt} + \mathbf{w}_\perp \cdot \nabla_\perp \hat{\mathbf{b}} \right), \quad (\text{V.23})$$

$$\frac{dw_\perp}{dt} = -\hat{\mathbf{e}}_\perp \cdot \left[ \left( \frac{D}{Dt} + \mathbf{w}_\perp \cdot \nabla_\perp \right) (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \right], \quad (\text{V.24})$$

where

$$\frac{D}{Dt} \doteq \frac{\partial}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \quad (\text{V.25})$$

is the Lagrangian time derivative in the parallel-streaming and  $\mathbf{E} \times \mathbf{B}$ -drifting frame. In (V.23) and (V.24) we find a mix of terms that are independent of  $\vartheta$  and dependent upon  $\vartheta$ . For example, grouping such terms in (V.23),

$$\frac{dv_\parallel}{dt} = \left\{ \frac{qE_\parallel}{m} + \mathbf{v}_E \cdot \frac{D\hat{\mathbf{b}}}{Dt} \right\} + \left\{ \mathbf{w} \cdot \left( \frac{D\hat{\mathbf{b}}}{Dt} + \mathbf{w}_\perp \cdot \nabla_\perp \hat{\mathbf{b}} \right) + \mathbf{w}_\perp \mathbf{v}_E \cdot \nabla_\perp \hat{\mathbf{b}} \right\}. \quad (\text{V.26})$$

To separate the two groups, we introduce the gyro-averaging procedure

$$\langle \dots \rangle_{\mathbf{R}} \doteq \frac{1}{2\pi} \oint d\vartheta (\dots), \quad (\text{V.27})$$

where the gyrophase integral is taken at fixed  $\mathbf{R}$ . The following identities are useful:

$$\langle \mathbf{w} \rangle_{\mathbf{R}} = w_{\parallel} \hat{\mathbf{b}}, \quad \langle \mathbf{w} \mathbf{w} \rangle_{\mathbf{R}} = w_{\parallel}^2 \hat{\mathbf{b}} \hat{\mathbf{b}} + \frac{w_{\perp}^2}{2} (\mathbf{I} - \hat{\mathbf{b}} \hat{\mathbf{b}}). \quad (\text{V.28})$$

Applying the gyro-average to (V.26) and using these identities yields

$$\langle \dot{v}_{\parallel} \rangle_{\mathbf{R}} = \left\{ \frac{qE_{\parallel}}{m} + \mathbf{v}_E \cdot \frac{D\hat{\mathbf{b}}}{Dt} \right\} + \underbrace{\left\{ \frac{w_{\perp}^2}{2} (\mathbf{I} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : \nabla_{\perp} \hat{\mathbf{b}} \right\}}_{= -\hat{\mathbf{b}} \cdot \nabla \ln B}$$

$$\boxed{\langle \dot{v}_{\parallel} \rangle_{\mathbf{R}} = \frac{qE_{\parallel}}{m} + \mathbf{v}_E \cdot \frac{D\hat{\mathbf{b}}}{Dt} - \frac{w_{\perp}^2}{2B} \hat{\mathbf{b}} \cdot \nabla B} \quad (\text{V.29})$$

So, guiding-center acceleration along field lines is driven by (1) parallel electric fields, (2) a fictitious force that accounts for boosting to the non-inertial frame of a varying  $\mathbf{v}_E$ , and (3) mirroring forces by parallel gradients in the magnetic-field strength. The interpretation of the second term is aided by noting that

$$\mathbf{v}_E \cdot \frac{D\hat{\mathbf{b}}}{Dt} = -\frac{D\mathbf{v}_E}{Dt} \cdot \hat{\mathbf{b}},$$

since  $\mathbf{v}_E \cdot \hat{\mathbf{b}} = 0$ . In the third term, you should recognize the combination  $w_{\perp}^2/2B$ .

Doing the same for  $w_{\perp} \dots$

$$\begin{aligned} \frac{dw_{\perp}}{dt} &= -(\hat{\mathbf{e}}_1 \cos \vartheta + \hat{\mathbf{e}}_2 \sin \vartheta) \cdot \left( v_{\parallel} \frac{D\hat{\mathbf{b}}}{Dt} + \frac{D\mathbf{v}_E}{Dt} \right) \\ &\quad - (\hat{\mathbf{e}}_1 \cos \vartheta + \hat{\mathbf{e}}_2 \sin \vartheta) \cdot (v_{\parallel} \mathbf{w}_{\perp} \cdot \nabla_{\perp} \hat{\mathbf{b}} + \mathbf{w}_{\perp} \cdot \nabla_{\perp} \mathbf{v}_E) \end{aligned}$$

$$\begin{aligned} \implies \langle \dot{w}_{\perp} \rangle_{\mathbf{R}} &= -(\langle \cos \vartheta \mathbf{w}_{\perp} \rangle_{\mathbf{R}} \cdot \nabla_{\perp} v_{\parallel} \hat{\mathbf{b}}) \cdot \hat{\mathbf{e}}_1 - (\langle \cos \vartheta \mathbf{w}_{\perp} \rangle_{\mathbf{R}} \cdot \nabla_{\perp} \mathbf{v}_E) \cdot \hat{\mathbf{e}}_1 \\ &\quad - (\langle \sin \vartheta \mathbf{w}_{\perp} \rangle_{\mathbf{R}} \cdot \nabla_{\perp} v_{\parallel} \hat{\mathbf{b}}) \cdot \hat{\mathbf{e}}_2 - (\langle \sin \vartheta \mathbf{w}_{\perp} \rangle_{\mathbf{R}} \cdot \nabla_{\perp} \mathbf{v}_E) \cdot \hat{\mathbf{e}}_2 \\ &= -\frac{w_{\perp}}{2} \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 : \nabla_{\perp} v_{\parallel} \hat{\mathbf{b}} - \frac{w_{\perp}}{2} \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 : \nabla_{\perp} \mathbf{v}_E \\ &\quad - \frac{w_{\perp}}{2} \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 : \nabla_{\perp} v_{\parallel} \hat{\mathbf{b}} - \frac{w_{\perp}}{2} \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 : \nabla_{\perp} \mathbf{v}_E \\ &= -\frac{w_{\perp}}{2} (\mathbf{I} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : \nabla v_{\parallel} \hat{\mathbf{b}} - \frac{w_{\perp}}{2} (\mathbf{I} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : \nabla \mathbf{v}_E \end{aligned}$$

$$\boxed{\langle \dot{w}_{\perp} \rangle_{\mathbf{R}} = \frac{v_{\parallel} w_{\perp}}{2B} \hat{\mathbf{b}} \cdot \nabla B - \frac{w_{\perp}}{2} (\mathbf{I} - \hat{\mathbf{b}} \hat{\mathbf{b}}) : \nabla \mathbf{v}_E} \quad (\text{V.30})$$

And, in a similar manner,

$$\boxed{\langle \dot{\vartheta} \rangle_{\mathbf{R}} = -\Omega - \hat{\mathbf{e}}_2 \cdot \frac{D\hat{\mathbf{e}}_1}{Dt} - \frac{v_{\parallel}}{2} \hat{\mathbf{b}} \cdot \nabla \times (v_{\parallel} \hat{\mathbf{b}} + \mathbf{v}_E)} \quad (\text{V.31})$$

But this one doesn't really matter – we'll only ever need the leading-order  $\dot{\vartheta} = -\Omega$ .

We can also go back and compute the  $\mathcal{O}(\epsilon)$  terms in  $\dot{\mathbf{R}}$  (see (V.15)), in order to see

the appearance of inhomogeneities in the evolution of the guiding center:

$$\begin{aligned}
\langle \dot{\mathbf{R}} \rangle_{\mathbf{R}} &= v_{\parallel} \hat{\mathbf{b}} + \mathbf{v}_E - \left\langle \frac{d\mathbf{v}_E}{dt} \times \frac{\hat{\mathbf{b}}}{\Omega} \right\rangle_{\mathbf{R}} + \left\langle \mathbf{w} \times \frac{d}{dt} \frac{\hat{\mathbf{b}}}{\Omega} \right\rangle_{\mathbf{R}} \\
&= v_{\parallel} \hat{\mathbf{b}} + \mathbf{v}_E - \frac{D\mathbf{v}_E}{Dt} \times \frac{\hat{\mathbf{b}}}{\Omega} + v_{\parallel} \hat{\mathbf{b}} \times \frac{D\hat{\mathbf{b}}}{Dt} + \left\langle \mathbf{w}_{\perp} \times \left( \mathbf{w}_{\perp} \times \nabla_{\perp} \frac{\hat{\mathbf{b}}}{\Omega} \right) \right\rangle_{\mathbf{R}} \\
&= v_{\parallel} \hat{\mathbf{b}} + \mathbf{v}_E - \frac{D\mathbf{v}_E}{Dt} \times \frac{\hat{\mathbf{b}}}{\Omega} + v_{\parallel} \hat{\mathbf{b}} \times \frac{D\hat{\mathbf{b}}}{Dt} \\
&\quad + \frac{w_{\perp}^2}{2} \left[ \hat{\mathbf{e}}_1 \times \left( \hat{\mathbf{e}}_1 \times \nabla \frac{\hat{\mathbf{b}}}{\Omega} \right) + \hat{\mathbf{e}}_2 \times \left( \hat{\mathbf{e}}_2 \times \nabla \frac{\hat{\mathbf{b}}}{\Omega} \right) \right]
\end{aligned}$$

$$\boxed{\langle \dot{\mathbf{R}} \rangle_{\mathbf{R}} = \left[ v_{\parallel} + \frac{w_{\perp}^2}{2\Omega} \hat{\mathbf{b}} \cdot (\nabla \times \hat{\mathbf{b}}) \right] \hat{\mathbf{b}} + \mathbf{v}_E + \frac{w_{\perp}^2}{2\Omega} \hat{\mathbf{b}} \times \nabla \ln B + \hat{\mathbf{b}} \times \left( v_{\parallel} \frac{D\hat{\mathbf{b}}}{Dt} + \frac{1}{\Omega} \frac{D\mathbf{v}_E}{Dt} \right)} \quad (\text{V.32})$$

Again, a reminder: every  $\hat{\mathbf{b}}$  and  $\mathbf{v}_E$  in these formulae are evaluated at  $(t, \mathbf{R})$ . From left to right, we have (1) parallel streaming (including an  $\mathcal{O}(\epsilon)$  correction to the parallel velocity), (2)  $\mathbf{E} \times \mathbf{B}$  drift, (3) grad- $B$  drift, (4) curvature drift, and (5) polarization drift.

## V.5. First adiabatic invariant

The equation for  $\langle \dot{w}_{\perp} \rangle_{\mathbf{R}}$ , (V.30), implies something special. Note that

$$\begin{aligned}
(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla \mathbf{v}_E &= \underbrace{\nabla \cdot \left( \frac{c\mathbf{E} \times \hat{\mathbf{b}}}{B} \right)}_{\text{use Faraday's law}} - \hat{\mathbf{b}} \cdot \left( \hat{\mathbf{b}} \cdot \nabla \frac{c\mathbf{E} \times \hat{\mathbf{b}}}{B} \right) \\
&= -\frac{\partial \ln B}{\partial t} - \underbrace{c\mathbf{E} \cdot \left( \nabla \times \frac{\hat{\mathbf{b}}}{B} \right) - \hat{\mathbf{b}} \cdot \left( \hat{\mathbf{b}} \cdot \nabla \frac{c\mathbf{E} \times \hat{\mathbf{b}}}{B} \right)}_{\text{use vector identities to expand}} \\
&= -\frac{\partial \ln B}{\partial t} - \frac{c\mathbf{E} \cdot (\nabla \times \hat{\mathbf{b}})}{B} - \underbrace{\frac{c\mathbf{E} \cdot (\hat{\mathbf{b}} \times \nabla \ln B)}{B} - \frac{c\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla (\mathbf{E} \times \hat{\mathbf{b}})}{B}}_{\text{use vector identities to rearrange}} \\
&= -\frac{\partial \ln B}{\partial t} - \underbrace{\frac{c\mathbf{E} \cdot (\nabla \times \hat{\mathbf{b}})}{B}}_{\text{write}} - \underbrace{\frac{c\mathbf{E} \times \hat{\mathbf{b}}}{B} \cdot \nabla \ln B}_{= \mathbf{v}_E} + \frac{c\mathbf{E}_{\perp} \cdot (\nabla \times \hat{\mathbf{b}})}{B} \\
&= -\frac{\partial \ln B}{\partial t} - \underbrace{\frac{cE_{\parallel} \hat{\mathbf{b}} \cdot (\nabla \times \hat{\mathbf{b}})}{B}}_{\text{is } \mathcal{O}(\epsilon) \text{ relative to other terms}} - \mathbf{v}_E \cdot \nabla \ln B \\
&= -\frac{\partial \ln B}{\partial t} - \mathbf{v}_E \cdot \nabla \ln B + \mathcal{O}(\epsilon)
\end{aligned}$$

And so (V.30) becomes

$$\begin{aligned}\langle \dot{w}_\perp \rangle_{\mathbf{R}} &= \frac{v_\parallel w_\perp}{2} \hat{\mathbf{b}} \cdot \nabla \ln B - \frac{w_\perp}{2} \left( -\frac{\partial \ln B}{\partial t} - \mathbf{v}_E \cdot \nabla \ln B \right) + \mathcal{O}(\epsilon) \\ &= \frac{w_\perp}{2} \left( \frac{\partial}{\partial t} + v_\parallel \hat{\mathbf{b}} \cdot \nabla + \mathbf{v}_E \cdot \nabla \right) \ln B + \mathcal{O}(\epsilon) \\ &= \frac{w_\perp}{2} \frac{D \ln B}{Dt} + \mathcal{O}(\epsilon),\end{aligned}$$

which implies

$$\boxed{\langle \dot{\mu} \rangle_{\mathbf{R}} = \mathcal{O}(\epsilon)} \tag{V.33}$$

where  $\mu \doteq w_\perp^2/2B(t, \mathbf{R})$ . In words, the magnetic moment  $\mu$  is constant on the time and length scales of the field variation. Its constantness is telling us that, on these time and length scales,  $\vartheta$  is an ignorable coordinate. (This property forms the basis of gyrokinetics.) More fundamentally,  $\mu$  conservation is telling us that plasmas are ‘diamagnetic’, that is, all particle-generated fluxes add to reduce the ambient field. The total change in  $B$  is proportional to the change in the perpendicular kinetic energy of the particle. The greater the plasma thermal energy, the more it excludes the magnetic field.

For a fluid element containing an ensemble of magnetized particles,  $\mu$  conservation implies that the thermal pressure perpendicular to the local magnetic field of that fluid element  $P_\perp \doteq \langle mw_\perp^2/2 \rangle \propto nB$ , where the angle brackets  $\langle \cdot \rangle$  denote the ensemble average. We’ll return to this important point later in the course.

## V.6. Adiabatic invariance

$\mu$  is one of several adiabatic invariants, which are related to the exactly conserved Poincaré invariants of classical mechanics. Adiabatic invariance is one of the most important concepts in the plasma physics of weakly collisional plasmas. The invariants emerge from the periodic motion induced by the magnetic field, and derive from the Hamiltonian action  $\oint \varphi \cdot d\mathbf{q}$  around a loop representing nearly periodic motion.  $\mu$  is the ‘first adiabatic invariant’ of plasma physics; the corresponding periodic motion is obviously the gyromotion of a particle about a magnetic field. The canonical momentum  $\varphi$  in this case is the particle’s angular momentum,  $mv_\perp \rho$ ; the angular variable  $\vartheta$  is the  $q$ , the conjugate coordinate. If the particle’s orbit changes slowly, either because  $\partial_t \ln B \ll \Omega$  or because the particle is drifting slowly into a region of varying field strength and/or geometry, then the action changes very little.<sup>7</sup> You might see a ‘simple’ derivation of  $\mu$  conservation in some textbooks, rather different from the guiding-center-theory approach

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<sup>7</sup>How little? [Kruskal \(1958, 1962\)](#) and [Northrop \(1963b\)](#) showed that  $\mu$  is conserved ‘to all orders’, meaning that, if  $\mu$  can be written as an expansion in the small parameter  $\epsilon$ ,  $\mu = \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \dots$ , then  $\Delta\mu \doteq \mu - \mu_0 = c_1 \exp(-c_2\epsilon)$ , where  $c_1$  and  $c_2$  are positive constants of order unity.

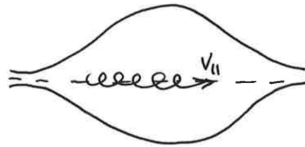


we've taken above. It runs something like this:

$$\begin{aligned}
 (\Delta\mu \text{ in one orbit}) &= \frac{\Delta(mw_{\perp}^2/2)}{B} - \mu \frac{\Delta B}{B} = \frac{1}{B} \int_0^{2\pi/\Omega} dt \frac{d}{dt} \left( \frac{1}{2}mw_{\perp}^2 \right) - \mu \frac{\Delta B}{B} \\
 &= \frac{1}{B} \int_0^{2\pi/\Omega} dt q\mathbf{w}_{\perp} \cdot \mathbf{E}_{\perp} - \mu \frac{\Delta B}{B} \\
 &= \frac{q}{B} \oint d\boldsymbol{\ell}_{\perp} \cdot \mathbf{E}_{\perp} - \mu \frac{\Delta B}{B} \\
 &= \mu \frac{\Delta B}{B} - \mu \frac{\Delta B}{B} = 0.
 \end{aligned}$$

The idea is that the electric field associated with the change in the magnetic field accelerates the particle, increasing its perpendicular energy in such a way that  $\mu$  is conserved.

A nice example of adiabatic invariance at work is magnetic mirroring. Imagine a magnetized particle trapped inside the potential well of a static magnetic bottle:



The energy of the particle is conserved,

$$\varepsilon = \frac{1}{2}mv_{\parallel}^2 + \frac{1}{2}mv_{\perp}^2 = \text{const},$$

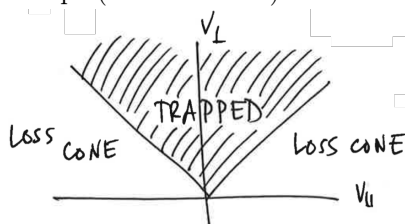
as is its magnetic moment,  $\mu = \text{const}$ . Thus, as the particle moves from its initial position where the magnetic-field strength is  $B_0$  into a region where the field strength is  $B$ , its parallel velocity, initially  $v_{\parallel 0}$ , must adjust according to these constraints:

$$\frac{1}{2}mv_{\parallel 0}^2 + \mu B_0 = \frac{1}{2}mv_{\parallel}^2 + \mu B = \varepsilon \implies v_{\parallel} = \pm \sqrt{\frac{2}{m}(\varepsilon - \mu B)} \tag{V.34}$$

With  $\varepsilon$  and  $\mu$  constant, this establishes a relationship between the parallel velocity of the particle and the local magnetic-field strength (at the particle's gyro-center): if  $B$  increases in the particle's frame,  $v_{\perp}$  must increase by  $\mu$  conservation, and  $v_{\parallel}$  must then decrease by energy conservation. If the particle encounters a strong enough magnetic field that  $v_{\parallel} \rightarrow 0$ , the particle is said to 'reflect' off of the strong-field region. The criterion for reflection is (V.34) with  $v_{\parallel} = 0$ :

$$\frac{1}{2}mv_{\parallel 0}^2 + \mu(B_0 - B) = 0 \implies \frac{v_{\parallel 0}}{v_{\perp 0}} \leq \sqrt{\frac{B}{B_0} - 1} \text{ for confinement} \tag{V.35}$$

This defines a critical pitch angle separated particles that are trapped inside the magnetic bottle from those that can escape (the 'loss cone'):



Collisions, which break  $\mu$ , would of course promote the leakage of particles out of the trapped region.

Now, what if the ends of the mirror were to move slowly?

## V.7. Second adiabatic invariant

Imagine a charged particle confined in a square-well potential:



Assume that the bounce time (i.e., the time required for the particle to transit the mirror, bounce, and return to its starting point) is much less than the time over which the ends of the mirror move. There will be an approximately conserved quantity,

$$\mathcal{J} \doteq \oint ds mv_{\parallel}, \quad (\text{V.36})$$

associated with the periodic bounce motion of the guiding center in the evolving mirror. This integral – the second adiabatic invariant – is taken over the ‘bounce orbit’ of the guiding center, with the differential  $ds$  oriented along the local magnetic-field direction and the limits being the turning points of the bounce orbit.<sup>8</sup> For example, if the mirror shrinks adiabatically, then  $v_{\parallel}$  increases.<sup>9</sup> Proving this is more involved, and  $\mathcal{J}$  is typically a less robust invariant than  $\mu$  (although it is of crucial importance for the persistence of the van Allen belts, by ensuring that precessing particles trapped in the Earth’s magnetic field return to their native field line after circumnavigating the Earth). If you’re interested in the finer details, consult [Northrop \(1963a, pg. 294\)](#).

## V.8. Third adiabatic invariant

There is yet another adiabatic invariant associated with the periodic motion of charged particles in a magnetic field, but it often receives much less attention than the first two because of its lesser utility. The reason is because the associated periodic motion is not as general as, say, a particle gyrating about a field line. In this case, the approximately conserved quantity

$$\mathcal{K} \doteq \oint dl \varphi_{\phi} \simeq e \oint dl A_{\phi} = e\Phi \quad (\text{V.37})$$

is the magnetic flux enclosed within a periodic orbit caused by cross-field drifts. (The drift velocity  $v_{\phi}$  is typically small compared to  $eA_{\phi}/c$ , thus the ‘ $\simeq$ ’ in (V.37).) If the particle orbit also involves bouncing between two turning points in a magnetic mirror, then the periodic orbit associated with the drift motion is to be evaluated at the ‘bounce center’ (just as  $\mathcal{J}$  is to be evaluated using an orbit of the guiding center). As with all adiabatic invariants, there is a comparison of time scales that must be done; here, it is between

<sup>8</sup>The canonical momentum here is technically  $mv_{\parallel} + eA_{\parallel}/c$ , but the latter (vector-potential) term representing the momentum associated with the electromagnetic field, once integrated over the bounce orbit, equals the total amount of magnetic flux enclosed by the orbit ( $= 0$ ).

<sup>9</sup>Note that both  $\mu$  and  $\mathcal{J}$  are of the form (energy)/(frequency). This is the general form of an adiabatic invariant. Think of  $E/\omega = \hbar$  (Einstein) or  $\oint p dq = nh$  (Sommerfeld). Einstein, at the Solray conference in 1911, said that this is the general form of an adiabatic invariant, and that this is what ought to be quantized.

the time scale on which the magnetic field varies and the period of the drift orbit. In the Earth's inner magnetosphere, the time for trapped particles with energies of  $\sim$ MeV to circumnavigate the Earth via their cross-field drifts is  $\sim$ 1 hr, and so any geomagnetic storms would interfere with  $\mathcal{K}$  conservation. Again, the separation of time scales and field geometry required for  $\mathcal{K}$  conservation is not particularly general, but it is important to bear in mind that particles like to keep the total magnetic flux constant within both their gyro-orbits ( $\mu$  conservation) and their drift orbits ( $\mathcal{K}$  conservation).

## V.9. Application: Magnetic pumping

Imagine a stationary, uniform, magnetically confined plasma whose thermal pressure  $P_0$  is initially isotropic, i.e.,  $P_{\perp}(0) = P_{\parallel}(0) = P_0$ , where  $P_{\perp} \doteq \langle mv_{\perp}^2/2 \rangle$  is the perpendicular pressure and  $P_{\parallel} \doteq \langle mv_{\parallel}^2 \rangle$  is the parallel pressure. (Again, the angle brackets  $\langle \cdot \rangle$  denote an ensemble average over all the particle constituting the plasma. Note that  $3P \doteq \langle mv^2 \rangle = P_{\parallel} + 2P_{\perp}$ .) Take the magnetic field to be uniform with strength  $B_0$ .

Slowly increase the strength of this field from  $B_0$  to  $B_1$ . 'Slowly' here means that the rate of increase is slow compared with the gyro-frequency of the trapped particles but fast compared with the rate at which collisions establish isotropy of the particle distribution function. Then, by adiabatic invariance, we have  $P_{\perp} = (B_1/B_0)P_0$  and  $P_{\parallel} = P_0$ . Now, wait. Eventually, energy-conserving Coulomb collisions will isotropize the temperature, so that  $P_{\perp} = P_{\parallel} = (1 + 2B_1/B_0)(P_0/3)$ . Once the system is well equilibrated, decrease the magnetic-field strength back to its initial value,  $B_0$ , again at a rate that is slow compared to the gyro-frequency of the trapped particles but fast compared with the collision frequency. Then  $P_{\perp} = (B_0/B_1)(1 + 2B_1/B_0)(P_0/3)$  and  $P_{\parallel} = (1 + 2B_1/B_0)(P_0/3)$ . Again, wait long enough for the temperatures to equilibrate. The final isotropic pressure is given by

$$P = \left[ \frac{2 + 5(B_1/B_0) + 2(B_1/B_0)^2}{9(B_1/B_0)} \right] P_0.$$

For  $B_1 = 2B_0$ , this is an  $\simeq$ 11% increase in the thermal energy of the plasma. Repeating this cycle 7 times more than doubles the thermal energy!

Questions to ponder: Where did this thermal energy come from? If you had increased/decreased the field strength on a time scale much longer than the collisional equilibration time, how would  $P$  have changed? Suppose this plasma were instead confined in a magnetic mirror reminiscent of that drawn in §V.7, whose length decreases and then increases by a factor of 2 in each cycle. Then what?

## REFERENCES

- BALBUS, STEVEN A. 1988 On Thermal Instability and Hydrostatic Equilibrium in Cooling Flows. *Astrophys. J.* **328**, 395.
- BALBUS, STEVEN A. & SOKER, NOAM 1989 Theory of Local Thermal Instability in Spherical Systems. *Astrophys. J.* **341**, 611.
- KRUSKAL, M. D. 1958 The Gyration of a Charged Particle. *Project Matterhorn Publications and Reports* .
- KRUSKAL, M. D. 1962 Asymptotic Theory of Hamiltonian and other Systems with all Solutions Nearly Periodic. *Journal of Mathematical Physics* **3**, 806–828.
- NORTHROP, T. G. 1963a Adiabatic Charged-Particle Motion. *Reviews of Geophysics and Space Physics* **1**, 283–304.
- NORTHROP, T. G. 1963b *The adiabatic motion of charged particles*. New York: Interscience Publishers.