Magnetohydrodynamic Instabilities, I.

Jeans, Schwarzschild, Parker, Rayleigh,

Velikhov-Chandrasekhar-Balbus-Hawley (magnetorotational)

But first...

$$\frac{d^2x}{dt^2} + \underbrace{2\nu}_{\substack{>0 \\ \text{damping}}} \frac{dx}{dt} + \underbrace{\Omega^2}_{(\text{freq.})^2} (x - \overbrace{x_0}^{\text{equilibrium position}}) = 0$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$x = x_0 + \underbrace{\xi}_{\text{displacement from equilibrium}} e^{-i\omega t}$$

$$\frac{d}{dt} \to -i\omega \qquad \left(\frac{d}{dt} e^{-i\omega t} = -i\omega e^{-i\omega t}\right)$$

$$(-i\omega)^2 \xi + 2\nu(-i\omega)\xi + \Omega^2 \xi = 0$$

$$\longrightarrow (-\omega^2 - i\omega 2\nu + \Omega^2)\xi = 0$$

$$\left(-\omega^2 - i\omega 2\nu + \Omega^2\right)\xi = 0$$

$$\rightarrow \omega^2 + i\omega 2\nu - \Omega^2 = 0$$

$$\boxed{\omega = -i\nu \pm \sqrt{\Omega^2 - \nu^2}}$$

real if $\Omega > \nu$; imaginary if $\Omega < \nu$

ignored initial conditions... this was only to compute frequency response to a displacement.

$$e^{-i\omega t} \rightarrow e^{-i(-i\nu)t} = e^{-\nu t}$$ damping if $\nu > 0$

Now suppose $\nu < 0$. $e^{-i\omega t} \rightarrow e^{\nu t}$ growing!

$$\boxed{\text{STABILITY CRITERION}: \quad \nu \geq 0}$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \Omega^2 \underbrace{\sin(\overbrace{x-x_0}^{=\xi})}_{|x-x_0| \ll x_0}\, = 0$$

$$\xi - \frac{\xi^3}{6} + \dots$$

To keep solutions $\sim e^{-i\omega t}$, need linear equations in time $\Longrightarrow$ small displacements in general

Can also run this argument in space...

$$\frac{d^2f}{dx^2} + (\dots)f = 0$$

$$g(x) \simeq g(x_0) + \left.\frac{dg}{dx}\right|_{x_0}(x-x_0)$$

To keep solutions $\sim e^{i\vec{k}\cdot\vec{r}}$, need linear equations in space $\Longrightarrow$ $k = \frac{2\pi}{\lambda} \gg \frac{1}{L_{\text{background}}}$

Looking at $\underline{\underline{\text{local}}}$ and $\underline{\underline{\text{linear}}}$ displacements.

variation of the background $\ll$ variation of waves/instabilities

small-amplitude displacements

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}$$

background flow

ideal MHD

no dissipation

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{u}$$

background gradients

$$\rho\frac{D\vec{u}}{Dt} = -\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{\vec{B}\cdot\vec{\nabla}}{4\pi}\vec{B} + \boxed{\rho\vec{g}}$$

$$\frac{D\vec{B}}{Dt} = \vec{B}\cdot\vec{\nabla}\vec{u} - \vec{B}\,\vec{\nabla}\cdot\vec{u}$$

$$\frac{P}{\gamma-1}\frac{D}{Dt}\ln\frac{P}{\rho^\gamma} = 0$$

$\vec{\nabla}\cdot\vec{g} = -4\pi G\rho$ if ρ is self-gravitating

① Jeans instability (non-magnetic)

$$\rho = \boxed{\rho_0} + \boxed{\delta\rho}$$

$$\vec{u} = 0 + \delta\vec{u}$$

$$P = P_0 + \delta P$$

uniform background

perturbations

$P_0 + P_1$

$P + \delta P$

Only retain terms linear in $\delta(\ldots)$:

(cont.) $$\frac{\partial}{\partial t}\delta\rho + \rho_0 \vec{\nabla}\cdot\delta\vec{u} = 0$$

(mom.) $$\rho_0 \frac{\partial}{\partial t}\delta\vec{u} = -\vec{\nabla}\delta P + \cancel{\delta\rho\,\vec{g}} + \rho_0\,\delta\vec{g}$$

"Jeans swindle"

(entr.) $$\frac{\partial}{\partial t}\left(\frac{\delta P}{P_0} - \gamma\frac{\delta\rho}{\rho_0}\right) = 0$$

$$\vec{\nabla}\cdot\delta\vec{g} = -4\pi G\,\delta\rho \qquad \Longrightarrow \delta \sim e^{-i\omega t + i\vec{k}\cdot\vec{r}}$$

$$\frac{\partial}{\partial t}\delta\rho + \rho_0 \vec{\nabla}\cdot\delta\vec{u} = 0$$

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$$\Downarrow$$

$$\frac{\partial}{\partial t} \to -i\omega$$

$$\vec{\nabla} \to i\vec{k}$$

$$\underbrace{\begin{bmatrix} - & - & \_\_ & \cdot \\ - & \_\_ & \_\_ & \\ - & \_\_ & \_\_ & \\ - & \_\_ & \_\_ & \end{bmatrix}}_{\omega, k} \begin{bmatrix} \delta\rho \\ \delta u_x \\ \delta u_y \\ \delta u_z \\ \delta P \end{bmatrix} = 0$$

$$\omega\left(\underline{\omega^2 - k^2 a^2 + 4\pi G \rho_0}\right) = 0$$

HAD $\frac{\delta P}{P_0} = \gamma \frac{\delta \rho}{\rho_0}$

$$\left[a^2 \equiv \frac{\gamma P_0}{\rho_0} \text{ is square of sound speed} \right]$$

Set $G=0$ (turn off gravity): $\omega^2 = k^2 a^2$

SOUND WAVES

But, w/ $G \neq 0$ ...

$$\boxed{\omega = \pm ka \underbrace{\sqrt{1 - \frac{4\pi G \rho_0}{k^2 a^2}}}}$$

can be imaginary if

$1 - \frac{4\pi G \rho_0}{k^2 a^2} < 0$

or $k^2 a^2 < 4\pi G \rho_0$.

STABLE IFF

$k^2 a^2 \geq 4\pi G \rho_0$

$k = 2\pi/\lambda$

longer wavelengths (higher k) have more mass enclosed w/in λ → more self-gravity. Going to short wavelengths increases ($-\vec{\nabla}\delta P$) pressure force (restoring).

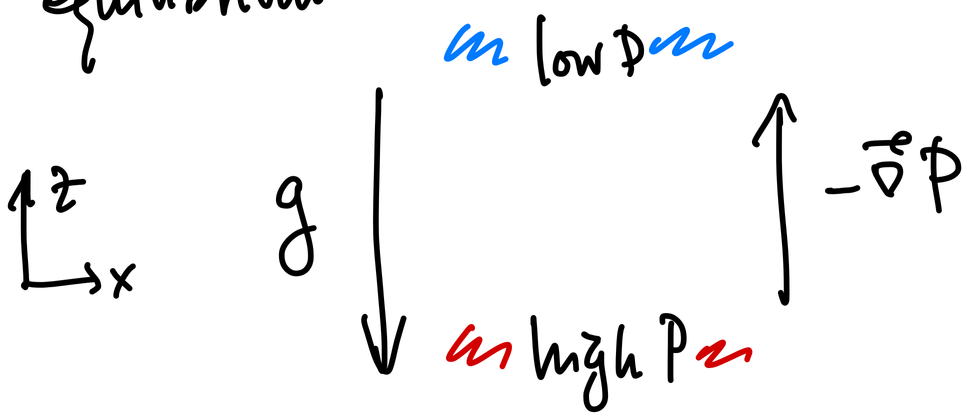
$$\lambda \leq \lambda_J \equiv a\sqrt{\frac{\pi}{G\rho}}$$

"Jeans length"

are stable.

Convective (Schwarzschild) instability

Suppose we have an atmosphere in hydrostatic equilibrium:



Balance: $0 = \rho \vec{g} - \vec{\nabla} P \longrightarrow \frac{1}{\rho_0}\frac{dP_0}{dz} = g$

Keeping it simple... $g = $ const.

$$\begin{aligned} \rho &= \underbrace{\rho_0(z)}_{} + \underbrace{\delta\rho(t,\vec{r})}_{} \\ \vec{u} &= 0 + \vec{\delta u}(t,\vec{r}) \\ P &= \underbrace{P_0(z)}_{\text{Background } z\text{-dependent}} + \underbrace{\delta P(t,\vec{r})}_{\text{perturbations}} \end{aligned}$$

linearize (keep terms linear in fluctuation amplitude)

$\Downarrow$

$\delta \sim e^{-i\omega t}$

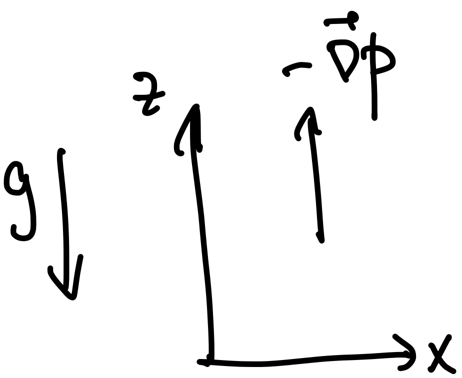
See notes

simple if $kH \gg 1$, where $H \equiv \left|\dfrac{d\ln p_0}{dz}\right|^{-1}$.

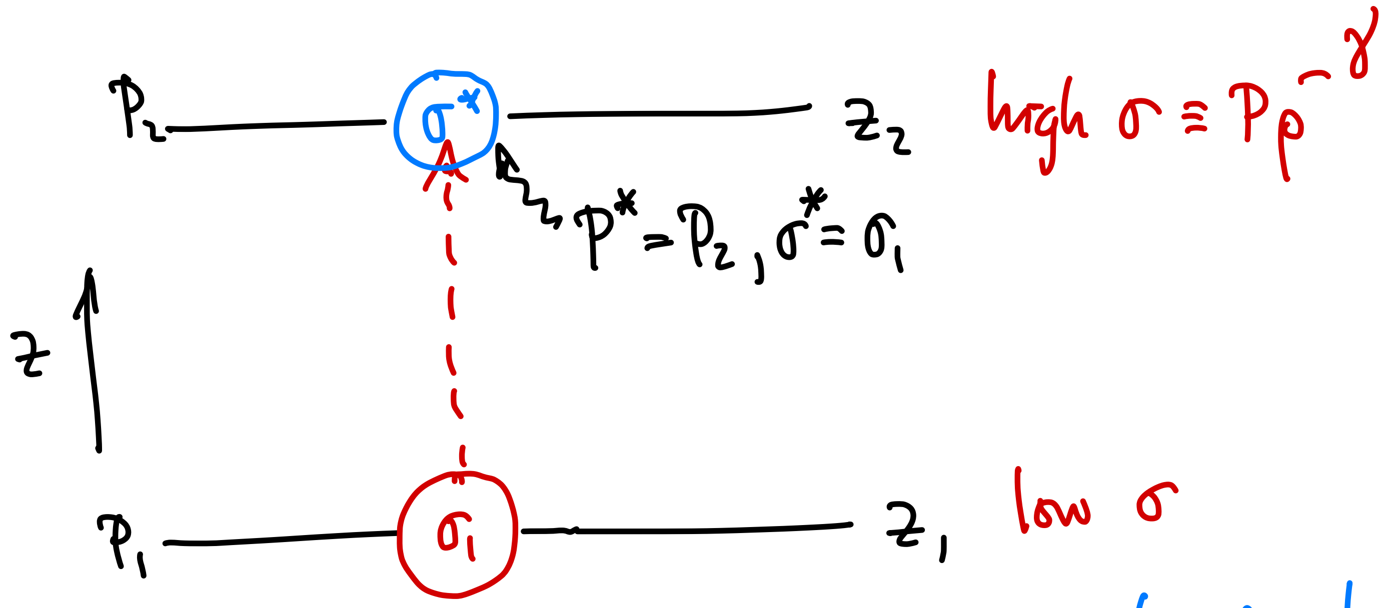
if fluctuations vary on scales $\ll$ the equilibrium

$$\boxed{\omega^2 = \frac{k_x^2}{k^2} N^2 \quad \text{where} \quad N^2 \equiv \frac{g}{\gamma}\,\frac{d\,\overbrace{\ln p\rho^{-\gamma}}^{\text{gradient of background entropy}}}{dz}}$$

Brunt-Väisälä frequency N



STABILITY CRITERION $\quad N^2 \geq 0 \quad$ (i.e., entropy must increase upwards)



What is the entropy of upwardly displaced fluid element, starts from σ_1? $\sigma^* = \sigma_1$

What changes? Density. If we lift fluid element up slow enough (Boussinesq approx.), sound waves will radiate and ensure fluid element is always in pressure balance with its surroundings.

Sps. $\sigma_1 > \sigma_2$

$$\rho^* = \left(\frac{p^*}{\sigma^*}\right)^{1/\gamma} = \left(\frac{P_2}{\sigma_1}\right)^{\frac{1}{\gamma}} \quad \text{vs.} \quad \rho_2 = \left(\frac{P_2}{\sigma_2}\right)^{\frac{1}{\gamma}}$$

$<$ unstable (keep rising)

Rotational stability

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}$$

$$= \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \Omega\frac{\partial}{\partial \varphi}$$

$$\vec{u} = \underbrace{\vec{v}}_{\text{measured velocity in rotating frame}} + R\Omega\,\hat{\varphi}$$

$$\frac{D\rho}{Dt} = -\rho\,\vec{\nabla}\cdot\vec{v}$$

$$\frac{Dv_R}{Dt} = \underbrace{2\Omega v_\varphi}_{\text{Coriolis}} + \underbrace{R\Omega^2 + \frac{v_\varphi^2}{R}}_{\text{centrifugal}} + f_R$$

$$\frac{Dv_\varphi}{Dt} = \underbrace{-\frac{\kappa^2}{2\Omega}v_R}_{\text{Coriolis + "boost"}} - R\frac{\partial\Omega}{\partial z}v_z - \frac{v_R v_\varphi}{R} + f_\varphi$$

$$\frac{Dv_z}{Dt} = f_z$$

$$\kappa^2 \equiv 4\Omega^2 + \frac{\partial\Omega^2}{\partial \ln R}$$

epicyclic frequency

Math... then,

ang. mom. momentum

$$\omega^2 = \frac{k_z^2}{k^2}\kappa^2$$

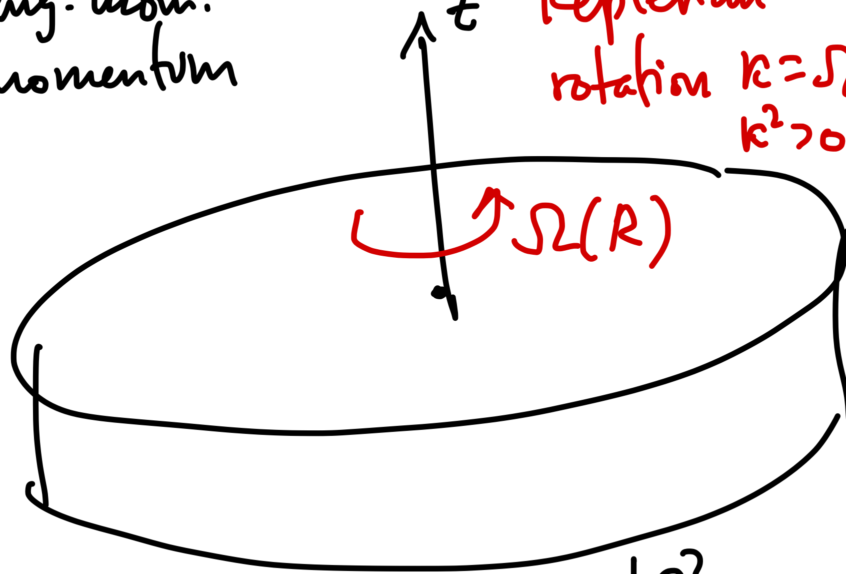
compare to

$$\omega^2 = \frac{k_x^2}{k^2} N^2$$

entropy gradient

Keplerian rotation $\kappa = \Omega$

$\kappa^2 > 0$.



$$\kappa^2 = 4\Omega^2 + \frac{d\Omega^2}{d\ln R}$$

$$= \frac{1}{R^3}\frac{d(\Omega^2 R^4)}{dR}$$

$$\equiv \frac{1}{R^3}\frac{d\ell^2}{dR}$$

w/ ℓ = specific angular momentum $= \Omega R$

This changes when you add magnetic fields (magneto rotational instability)

... $\vec{B}$ field tension breaks angular momentum conservation.

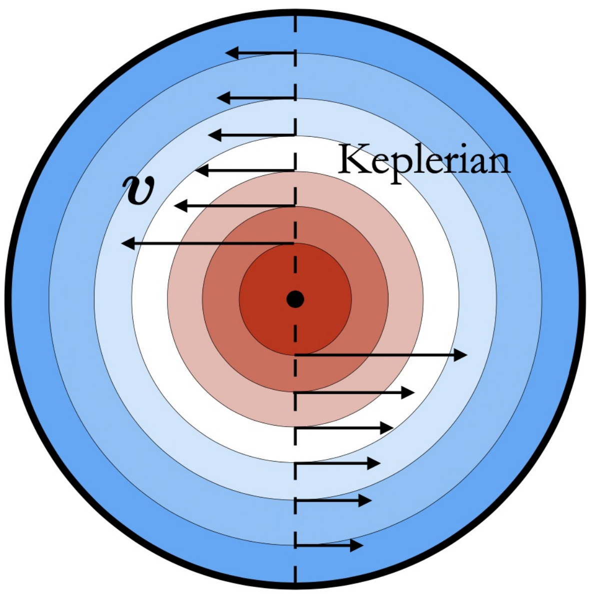
$$\omega^2 = \kappa^2 \longrightarrow d\Omega^2/d\ln R$$

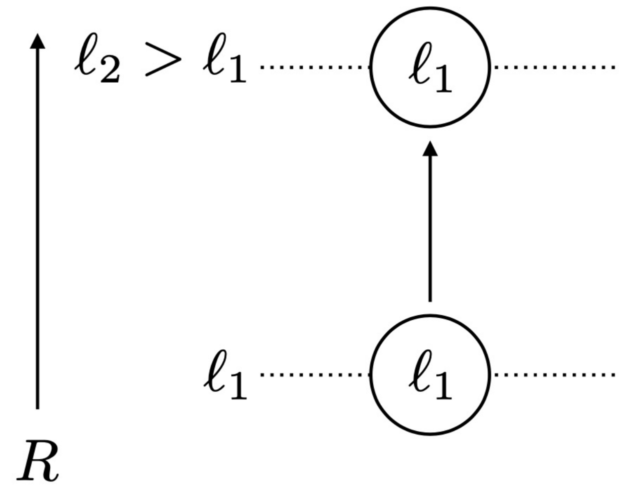
angular momentum, $\ell = \Omega R^2$:

$\Omega \propto R^{-3/2} \qquad \ell \propto R^{1/2}$

rotationally stable, since $\dfrac{\mathrm{d}\ell^2}{\mathrm{d}R} \geq 0$

Rayleigh criterion





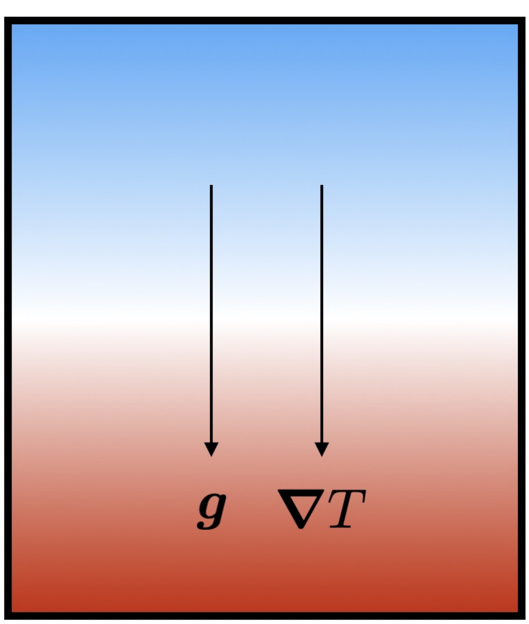
entropy, $\sigma = TP^{-2/5}$:

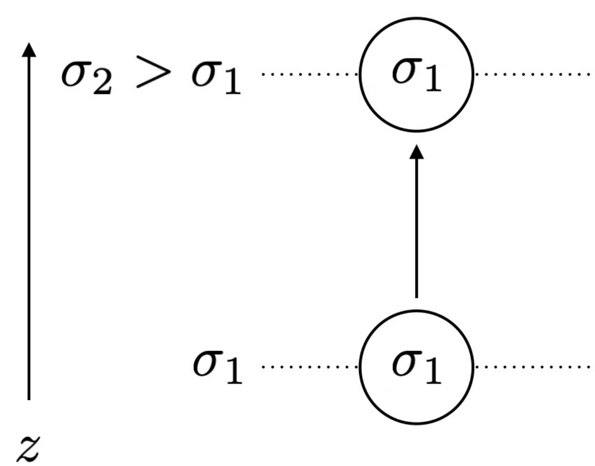
$T \propto z^{-1} \qquad \sigma \propto z^{1/3}$

entropy, $\sigma = TP^{-2/5}$:

convectively stable, since $\dfrac{\mathrm{d}\sigma}{\mathrm{d}z} \geq 0$

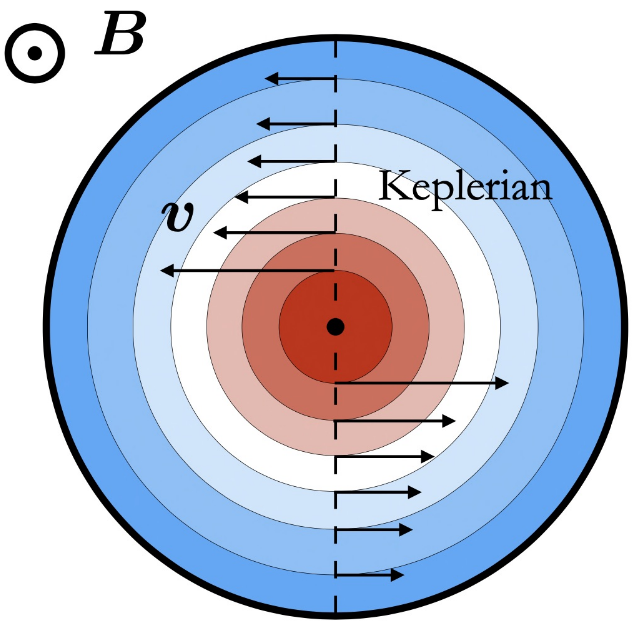
Schwarzschild criterion





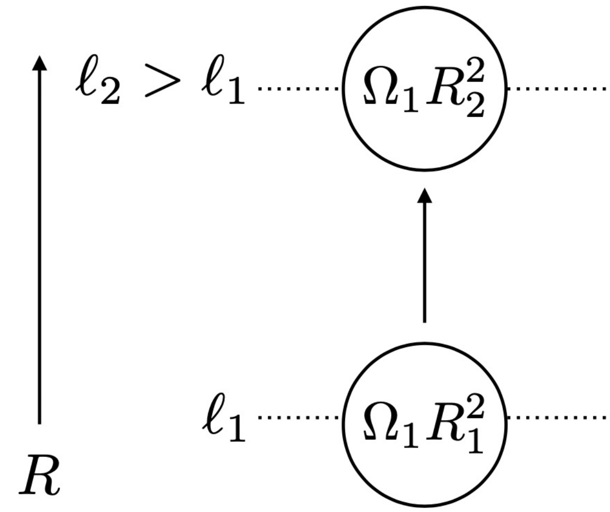
angular momentum, $\ell = \Omega R^2$:

$\Omega \propto R^{-3/2}$ $\qquad \ell \propto R^{1/2}$



rotationally *unstable*, since $\dfrac{\mathrm{d}\Omega^2}{\mathrm{d}R} < 0$

if magnetically tethered, $\Omega \approx$ const



(following Balbus 2001, *ApJ*)