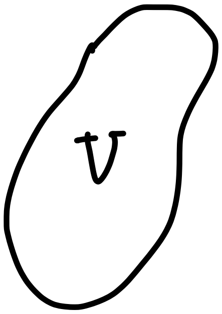
Fundamentals of fluid dynamics



volume of fluid, which is large enough to contain many particles but small enough so that its aspects have a unique value over that volume $\longrightarrow$ "fluid element"

Collection of particles: $(t, \underbrace{\vec{r}, \vec{v}}_{\text{6D}})$ (x, y, z, v_x, v_y, v_z)

Collection of fluid elements: $(t, \underbrace{\vec{r}}_{\text{3D}})$

6D $\rightarrow$ 3D! Savings! Pay the price by losing some information. In some systems, lost information doesn't matter ("collisional"); in other systems ("collisionless"), a fluid description is not rigorously valid.

Mass density $\equiv \rho$ $\leftarrow$ units g/cm^3

Fluid velocity $\equiv \vec{u}$ (u_x, u_y, u_z) $\leftarrow$ units cm/s

Pressure $\equiv P$ (scalar) $\leftarrow$ units $ergs/cm^3$

Need equations to evolve $\rho, \vec{u}, P \ldots$

Conservation laws:
1. Mass is conserved
2. $\vec{F} = m\vec{a}$
3. Entropy is conserved $\leftarrow$ "ideal"

1. Mass conservation $\longrightarrow$ "Continuity equation"

arbitrary volume

$\rho(t, \vec{r})$

$$M_{in} = \int_V dV \, \rho$$

V $\leftarrow$ fixed volume

$$\frac{dM_{in}}{dt} = ?$$

rate @ which mass flows in — rate @ which mass flows out

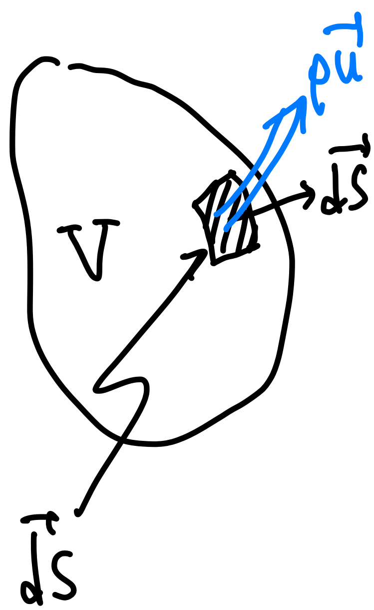
$$\frac{dM_{in}}{dt} = ?$$

$\underbrace{\text{rate @ which mass flows in} - \text{rate @ which mass flows out}}$

$$-\oint_{\partial V} d\vec{S}\cdot \rho\vec{u}$$

∂V = surface

$\rho\vec{u}$: flux of material



$$M_{in} = \int dV \rho$$

$$\frac{dM_{in}}{dt} = \int_V dV \boxed{\frac{\partial \rho}{\partial t}} = -\oint_{\partial V} d\vec{S}\cdot \rho\vec{u}$$

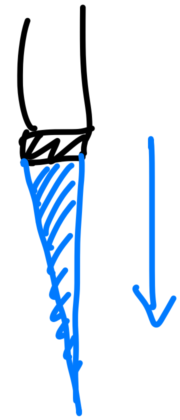
(by div. thm.)

$$= \boxed{-}\int_V dV \boxed{\vec{\nabla}\cdot(\rho\vec{u})}$$

arbitrary!

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0}$$

Continuity equation



falling water shrinks cross sectional A

② $\vec{F} = m\vec{a} \longrightarrow \vec{f} = \rho\vec{a}$

$\times$

$$\frac{\partial \vec{u}}{\partial t} = \ldots$$

Imagine yourself inside a fluid element... forces you experience are in that frame of reference, which might be moving.

$$\left.\frac{\partial \vec{u}}{\partial t}\right|_r$$

@ fixed r! WANT @ fixed fluid element.

something is fixed w/ partial derivative!

Instead,

$$\left.\frac{D\vec{u}}{Dt}\right| = \underbrace{\frac{\partial}{\partial t}\vec{u}}_{\text{lab frame}} + \underbrace{\frac{d\vec{r}}{dt}\cdot\vec{\nabla}\vec{u}}_{\text{boost}} = \left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\vec{u}$$

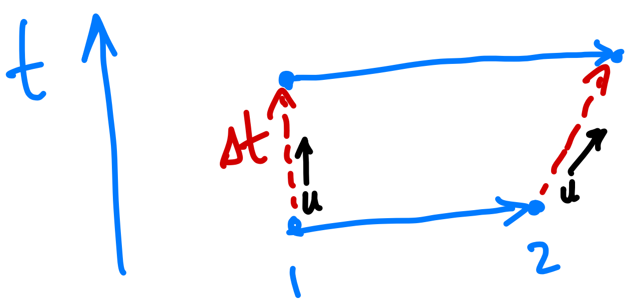
@ fixed fluid element $\vec{r}(t)$; $\frac{d\vec{r}}{dt} = \vec{u}$

$$\frac{D}{Dt}\vec{u} = \frac{\partial}{\partial t}\vec{u} + \underbrace{\vec{u}\cdot\vec{\nabla}}\,\vec{u}$$

$$u_x \frac{\partial}{\partial x}\vec{u} + u_y \frac{\partial}{\partial y}\vec{u} + u_z \frac{\partial}{\partial z}\vec{u}$$

Lagrangian derivative vs. Eulerian derivative

(comoving material convective) (fixed $\vec{r}$)



$$\vec{r}(t+\Delta t) - \vec{r}(t)$$

$$\approx (\vec{\delta r}\cdot\vec{\nabla})\,\vec{u}$$

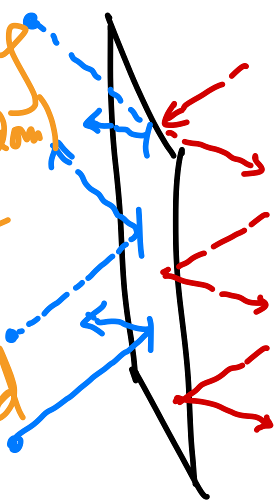
$$\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \underbrace{\frac{\vec{\delta r}}{\Delta t}}_{\vec{u}}\cdot\vec{\nabla}\,\vec{u}$$

$$\rho \frac{D\vec{u}}{Dt} = \rho\vec{g} + \vec{f}_{ext} - \vec{\nabla} P$$

"ma" gravity Lorentz + spoon + ... pressure gradients

Assume thermal (random) motion are fully randomized

Good for a collisional gas.



$P = \frac{F}{A}$

$$\overset{\text{right}}{P\left(t, x - \frac{\Delta x}{2}, y, z\right) dy\, dz}$$

$$\underset{\text{left}}{- P\left(t, x + \frac{\Delta x}{2}, y, z\right) dy\, dz}$$

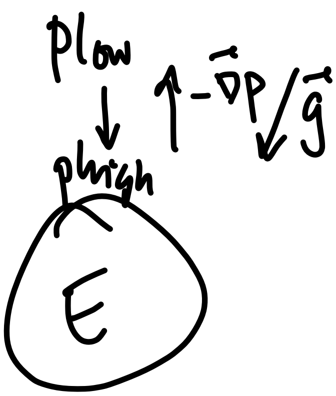
$$\xrightarrow[\Delta x \ll 1]{} - \frac{\partial P}{\partial x} dx\, dy\, dz$$

$\Downarrow$ promote to 3D

$$-\vec{\nabla} P \quad dV$$

this did not have to be a scalar

$$-\vec{\nabla} \cdot \overleftrightarrow{P}$$



$$P = \frac{\rho k_B T}{m} = n k_B T \quad \text{(ideal gas)}$$

$$\boxed{\rho \frac{D\vec{u}}{Dt} = \rho\vec{g} - \vec{\nabla} P + \text{forces} \ldots} \quad \text{Force equation}$$

Momentum density: $\rho\vec{u}$ $\quad \frac{\text{mass} \times \text{velocity}}{\text{volume}}$

$$\rho \frac{D\vec{u}}{Dt} = \rho\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\vec{u} + \Big[\underbrace{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u})}_{=0}\Big]\vec{u}$$

$$= \frac{\partial}{\partial t}(\rho\vec{u}) + \Big[\underbrace{\vec{\nabla}\cdot(\rho\vec{u}\,\vec{u})}_{\Sigma_i \frac{\partial}{\partial x_i}[\rho u_i u_j]}\Big]_j$$

$$\boxed{\text{Momentum eqn.:} \quad \frac{\partial}{\partial t}(\rho\vec{u}) + \vec{\nabla}\cdot(\rho\vec{u}\,\vec{u}) = \rho\vec{g} - \vec{\nabla}P + \ldots}$$

③ entropy conservation (or, first law of thermodynamics)

$$\text{internal energy density} \equiv e \equiv \frac{P}{\gamma - 1}.$$

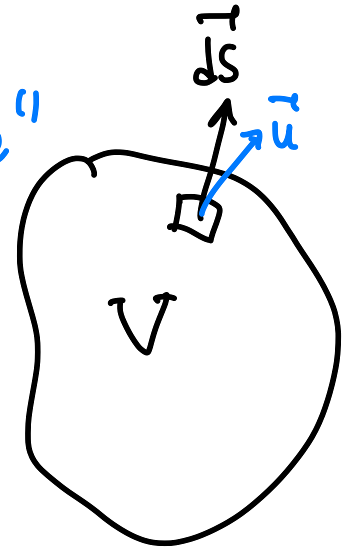
ratio of specific heats $= 5/3$

$$\boxed{\left.\frac{\partial e}{\partial t}\right|_r + \underbrace{\vec{\nabla}\cdot(e\vec{u})}_{\text{transport}} = -P\,\vec{\nabla}\cdot\vec{u}}$$

transport

(energy flux through a surface)

"p dV work"



$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\ln\frac{P}{\rho^\gamma} = 0$$

$$\frac{1}{P}\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)P - \frac{\gamma}{\rho}\underbrace{\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\rho}_{-\rho\vec{\nabla}\cdot\vec{u}} = 0$$

cont. eqn.

$$S = k_B \frac{1}{\gamma - 1}\ln\frac{P}{\rho^\gamma}$$

ideal gas

$$\boxed{\frac{Ds}{Dt} = 0}$$

$$\frac{1}{P}\left(\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}\right)P - \frac{\gamma}{\rho}\underbrace{\left(\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}\right)\rho}_{-\rho\vec{\nabla}\cdot\vec{u}} = 0$$

cont. equ.

$e = P/(\gamma-1)$ Multiply by $\frac{P}{\gamma-1}$ ...

$$\underbrace{\left(\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}\right)e}_{\frac{\partial e}{\partial t}+\vec{\nabla}\cdot(e\vec{u})-e\vec{\nabla}\cdot\vec{u}} + \gamma e\,\vec{\nabla}\cdot\vec{u} = 0$$

if self-gravity

$$\boxed{\vec{\nabla}\cdot\vec{g} = -4\pi G\rho}$$

$$\downarrow$$

$$\boxed{\frac{\partial e}{\partial t}+\vec{\nabla}\cdot(e\vec{u}) + (\gamma-1)e\,\vec{\nabla}\cdot\vec{u} = 0.}$$

ρ

$$\boxed{\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{u}) = 0}$$

$$\boxed{\begin{aligned}\rho\frac{D\vec{u}}{Dt} &= \frac{\partial(\rho\vec{u})}{\partial t}+\vec{\nabla}\cdot(\rho\vec{u}\vec{u}) \\ &= \rho\vec{g}-\vec{\nabla}P+\ldots\end{aligned}}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

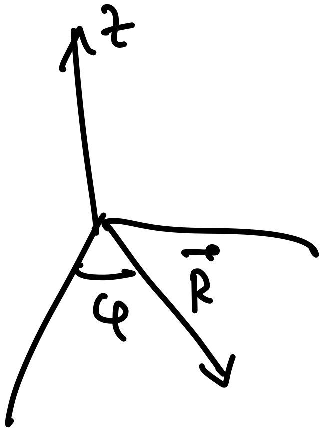
"BAC-CAB RULE"

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla}\cdot\vec{A}) + \vec{A}(\vec{\nabla}\cdot\vec{B})$$

Be careful! curvilinear coordinates...

$$(R,\varphi,z): \quad \frac{\partial \hat{R}}{\partial\varphi} = \hat{\varphi} \qquad \frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{R} \qquad \nabla_\varphi = \frac{1}{R}\frac{\partial}{\partial\varphi}$$



$$\vec{u}\cdot\vec{\nabla}\vec{u} = \vec{u}\cdot\vec{\nabla}(u_i\hat{e}_i)$$

$$= \hat{e}_i(\vec{u}\cdot\vec{\nabla}u_i)$$

$$+ u_i(\vec{u}\cdot\vec{\nabla}\hat{e}_i)$$

$$\text{Cylindricals} = u_R\,\underbrace{\vec{u}\cdot\vec{\nabla}\hat{R}}_{\hat{\varphi}} + u_\varphi\,\underbrace{\vec{u}\cdot\vec{\nabla}\hat{\varphi}}_{-\hat{R}} + \cancel{u_z\,\vec{u}\cdot\vec{\nabla}\hat{z}}^{\,0}$$

$$= (u_R u_\varphi/R)\,\hat{\varphi} + (-u_\varphi^2/R)\,\hat{R}$$

go to a rotating frame...

$$\vec{u} = R\Omega\hat{\varphi} + \vec{v}$$

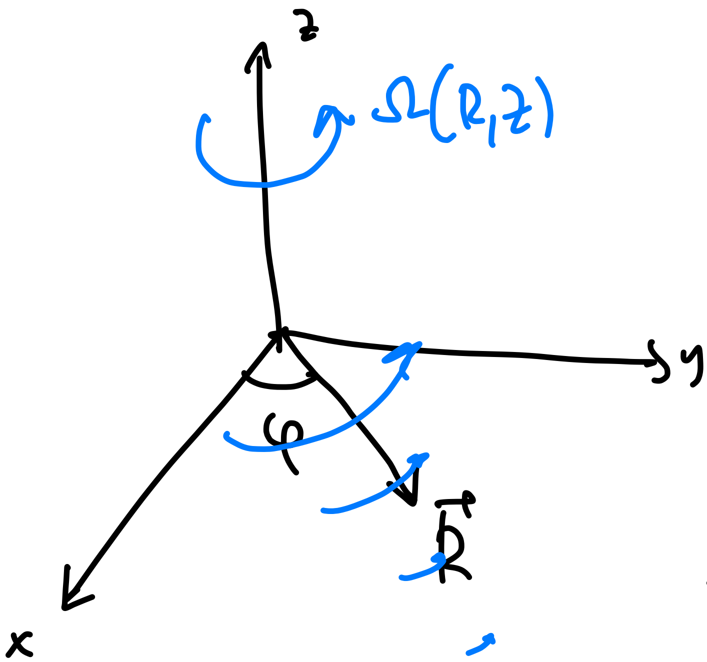
angular velocity

everything else

Compute $\vec{u}\cdot\vec{\nabla}\vec{u}$... Assume $\Omega = \Omega(R,z)$.

$$\left[\left(\vec{v} + R\Omega\hat{\varphi}\right)\cdot\vec{\nabla}\right]\left(\vec{v} + R\Omega\hat{\varphi}\right)$$

$$= \Big[\underbrace{\left(\vec{v}\cdot\vec{\nabla} + \Omega\frac{\partial}{\partial\varphi}\right)v_i}_{\text{advection by } \vec{v} + R\Omega\hat{\varphi}}\Big]\hat{e}_i$$

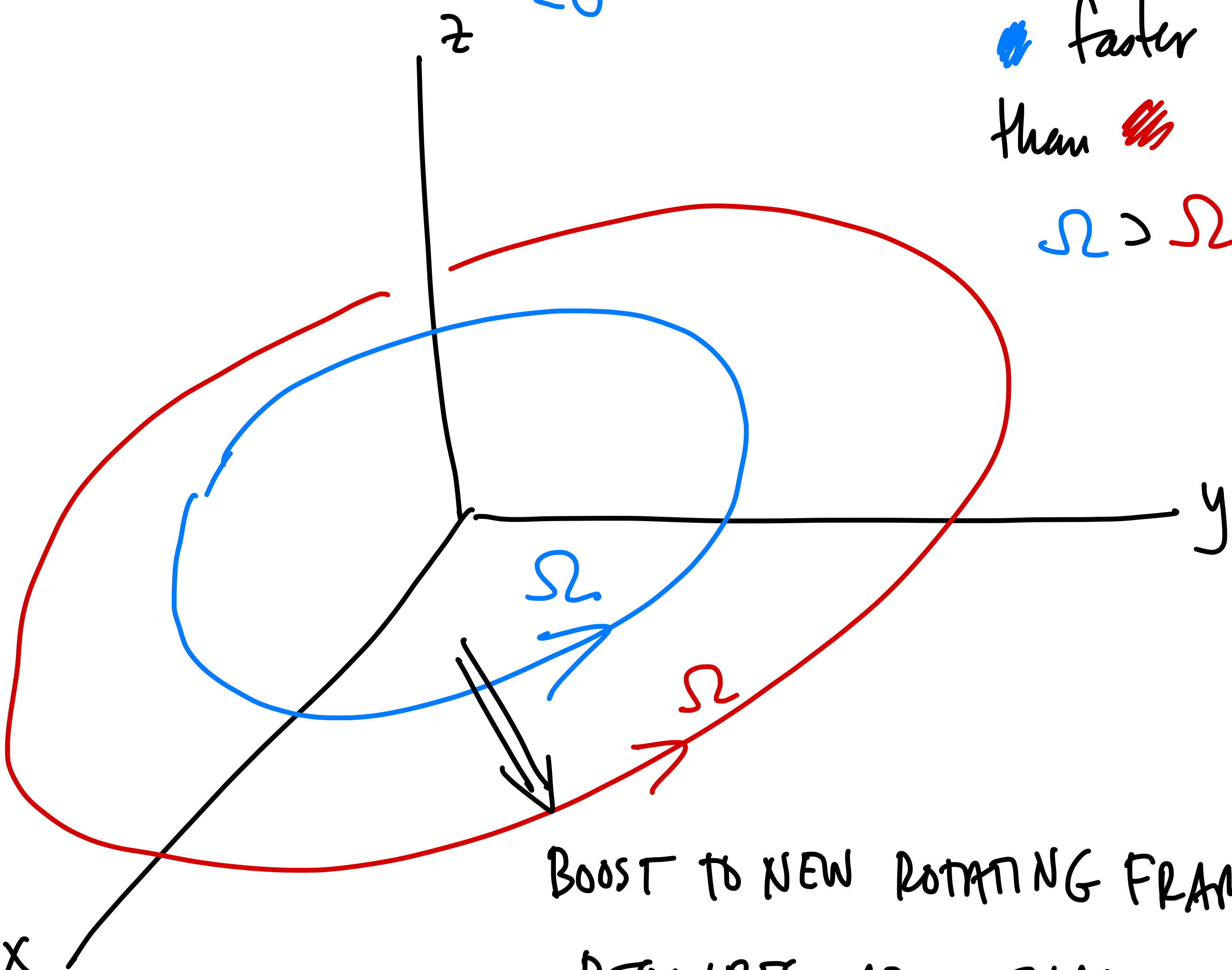


$$+\Big[\underbrace{2\Omega\hat{z}\times\vec{v}}_{\text{Coriolis}} - \underbrace{R\Omega^2\hat{R}}_{\text{centrifugal}} + \underbrace{R\hat{\varphi}\left(\vec{v}\cdot\vec{\nabla}\right)\Omega}_{\text{BOOST}}\Big]$$

$$+\Big[\underbrace{\frac{v_R v_\varphi}{R}\hat{\varphi}}_{\text{Coriolis}} - \underbrace{\frac{v_\varphi^2}{R}\hat{R}}_{\text{centrifugal}}\Big]$$

$$R\hat{\varphi}\,(\vec{v}\cdot\vec{\nabla})\,\Omega$$

$$= R\hat{\varphi}\left(v_R \underbrace{\frac{\partial\Omega}{\partial R}}_{<0} + v_z \frac{\partial\Omega}{\partial z}\right)$$



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