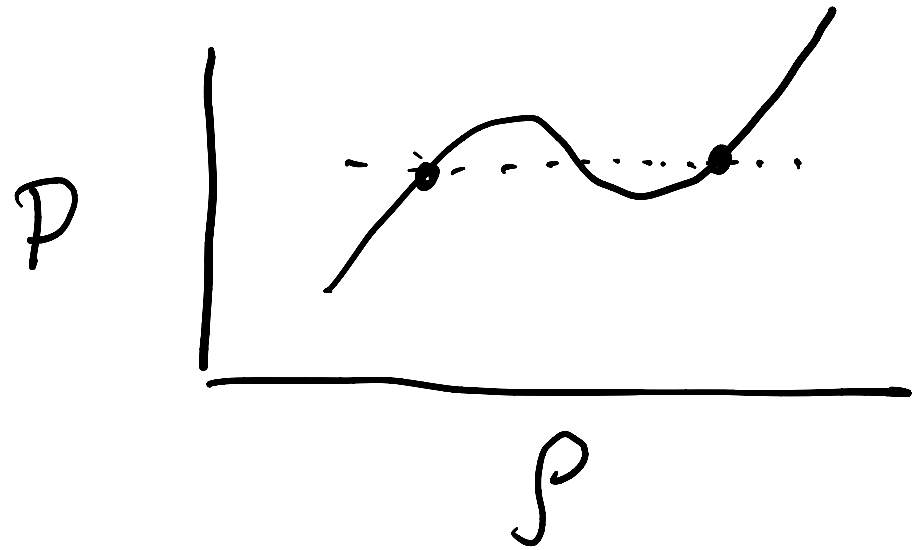
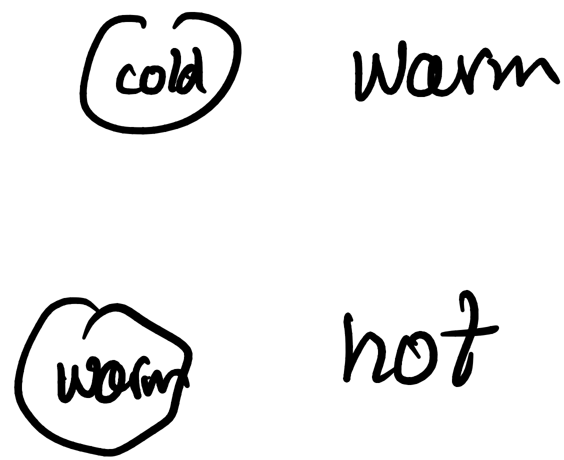
KH & RT instabilities

interfaces: equilibrium at two diff. T, n
same P





grav ≡ acceleration

(acceleration for steep dropoff in density equiv. to inward gravity)

ρ_{II} $\quad V_{II} \to V$

$\vec{B}_0 \to$ $\quad \downarrow g$ $\quad$ $z \uparrow$, $\to x$

ρ_I $\quad V_I \to 0$

assuming incompressible

$$\partial_t \to -i\omega$$

$$\partial_x \to i k_x$$

pert $\propto e^{i(kx - \omega t)}$

$$u_{1y} = 0$$

$$B_{1y} = 0$$

$$U = \begin{cases} V & z > 0 \\ 0 & z < 0 \end{cases}$$

can show (exercise):

$$\frac{\vec{B}_1}{B_0} = \frac{-k_x \vec{u}_1}{\omega - k_x U}$$ induction

$$u_{1x} = \frac{i}{k_x} \partial_z u_{1z}$$ mass cons. $\nabla \cdot \vec{u}_1 = 0$

$$\rho_0(\omega - k_x U)\vec{u}_1 = k_x P_1 \hat{x} - i\partial_z P_1 \hat{z}$$

$$+ \frac{\rho_0 V_A^2 k_x}{\omega - k_x U}(k_x u_{1z} + i\partial_z u_x)\hat{z}$$

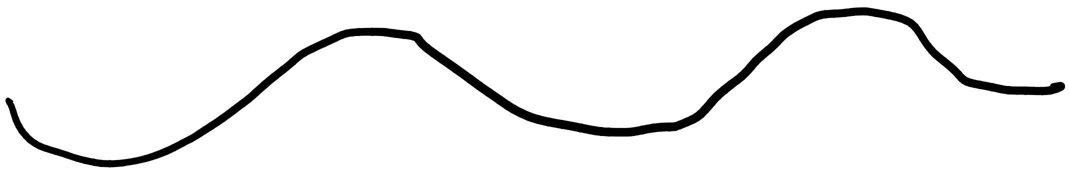
momentum eq

Apply ∂_z to $\hat{x}$ eq
and combine with $\hat{z}$ eq
to eliminate P terms

Away from interface, find that

$$\left[(\omega - k_x U)^2 - V_A^2 k_x^2\right]\left[k_x^2 u_{1z} - \partial_z^2 u_{1z}\right] = 0$$

$$u_{1z} = \begin{cases} A_{II}\, e^{-k_x z} & \text{(upper) } z>0 \\ A_I\, e^{k_x z} & \text{(lower) } z<0 \end{cases}$$



Lagrangian displacement of interface

δz **same** on both sides

$$(\partial_t + U\cdot\nabla)\,\delta z = u_{1z}$$

$$(-i\omega + ik_x U)\,\delta z = u_{1z}$$

$$\delta z\Big|_-^+ = \frac{u_{1z}}{-i\omega + ik_x U}\Bigg|_-^+ = 0$$

$$u_{1z} = A \begin{cases} (\omega - k_x V)\, e^{-k_x z} & \text{upper} \\ \omega\, e^{k_x z} & \text{lower} \end{cases}$$

Integrate momentum eq. across interface from $z = 0^-$ to $z = 0^+$

$$\rho_0 \frac{(\omega - k_x U)}{k_x^2} \partial_z u_{1z} \Big|_-^+ = \frac{B_0^2}{4\pi} \frac{\partial_z u_{1z}}{(\omega - k_x U)} \Big|_-^+$$

ρ_I or ρ_{II}

to obtain dispersion relation

$$\boxed{\rho_I \, \omega^2 + \rho_{II} (\omega - k_x V)^2 = \frac{B_0^2}{4\pi} k_x^2}$$

Dispersion relation for KH instab.

$$\omega = \frac{\rho_{II}}{\rho_I + \rho_{II}} V k_x \pm \left[\frac{B_0^2}{2\pi} \frac{k_x^2}{\rho_I + \rho_{II}} - \frac{\rho_I \rho_{II}}{(\rho_I + \rho_{II})^2} k_x^2 V^2 \right]^{1/2}$$

aside: $V = 0$ $\quad \omega = V_A k_x$

$\rho_I = \rho_{II}$

Unmagnetized:

$$\omega_i = \frac{\sqrt{\rho_I \rho_{II}}}{\rho_I + \rho_{II}} k_x V$$

fastest growing at large k_x small λ

growth higher @ larger V

ρ_I and ρ_{II} same

If $\rho_{II} \gg \rho_I$ $\omega_i \sim \sqrt{\frac{\rho_I}{\rho_{II}}}\, kV$

Magnetized:

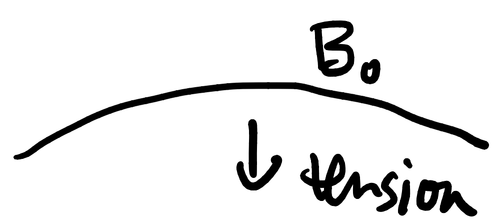
Magnetic tension stabilizes

Only have growth

$$\frac{B_0^2}{2\pi}\,\frac{1}{\rho_I+\rho_{II}} < \frac{\rho_I\,\rho_{II}}{(\rho_I+\rho_{II})^2}\,V^2$$

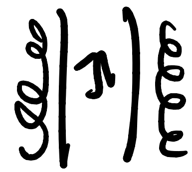
$$\frac{B^2}{2\pi\,\rho_{min}} < V^2$$

Why does KH happen in the first place?

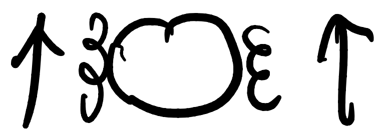


Applications

- jet → entrainment of ambient gas



- cloud in wind or falling



destroy cloud via mixing

- 

superbuble mixing at interfaces

If gravity had been included

$$\rho_I (\omega - k_x V_I)^2 + \rho_{II} (\omega - k_x V_{II})^2 = g k_x (\rho_I - \rho_{II}) + \frac{B_0^2}{2\pi} k_x^2$$

If $\rho_I > \rho_{II}$ stabilizing gravity

$\rho_I < \rho_{II}$ destabilizing

Rayleigh - Taylor instability

B ~~stabilizes~~ RT if

$$\frac{(\rho_{II} - \rho_I) g}{B_0^2 / 2\pi} < k_x$$

Energy & Equilibrium in MHD system

Magnetic Virial theorem

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0$$

$$\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u}\vec{u} + P \overleftrightarrow{I} - \overleftrightarrow{M}) = -\rho \nabla \Phi$$

$$\overleftrightarrow{M} = \frac{1}{4\pi} (\vec{B}\vec{B} - \tfrac{1}{2} B^2 \overleftrightarrow{I})$$

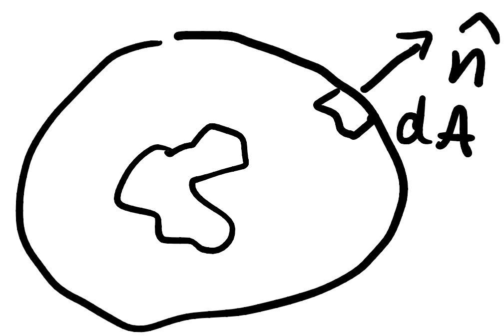
start with:

$$I = \int r^2 \rho \, d^3x$$

generalized moment of inertia

$$r = |\vec{x}|$$

$$r^2 = x^2 + y^2 + z^2$$



$$\ddot{I} = \int r^2 \partial_t^2 \rho \, d^3x \xrightarrow{\text{continuity eq}} = -\int r^2 \, \nabla\cdot\left(\partial_t(\rho\vec{u})\right) d^3x$$

Integrate by parts:

$$= \int \partial_t(\rho\vec{u}) \cdot \underbrace{\nabla r^2}_{2\vec{x}} \, d^3x - \oint r^2 \partial_t(\rho\vec{u}) \cdot \hat{n} \, dA$$

$$\frac{1}{2}\ddot{I} = \int \Big[-\nabla\cdot\big(\underbrace{\rho\vec{u}\vec{u} + P\overleftrightarrow{I} - \overleftrightarrow{M}}_{\overleftrightarrow{T}}\big) - \underbrace{\rho\nabla\Phi}_{\text{gravity term}} \Big] \cdot \vec{x} \, d^3x$$

$$- \frac{1}{2}\oint r^2 \partial_t(\rho\vec{u}) \cdot \hat{n} \, dA$$

Gravity term

$$-\int \rho \nabla \Phi \cdot \vec{x} \, d^3x$$

exercise:

$$= \quad E_G = -\frac{1}{2} G \iint \frac{\rho(\vec{x}) \rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \, d^3x \, d^3x'$$

$$= \frac{1}{2} \int \rho(\vec{x}) \, \Phi(\vec{x}) \, d^3x$$

$$\mathcal{I}_T = -\int (\nabla \cdot \overleftrightarrow{T}) \cdot \vec{x} \, d^3x$$

In index notation, where repeated indices
-has implied sum (over i and j)

$$= -\int d_i (T_{ij}) \, x_j \, d^3x$$

integrate by parts:

$$= -\int d_i (T_{ij} x_j) - T_{ij} d_i x_j \, d^3x$$

$$= -\oint \vec{x} \cdot \overleftrightarrow{T} \cdot \hat{n} \, dA + \int \underbrace{T_{ij} \, \delta_{ij}}_{\text{trace}(\overleftrightarrow{T})} d^3x$$

$$\mathrm{tr}(\overleftrightarrow{T}) = \rho|\vec{u}|^2 + 3P - \frac{1}{4\pi}\underbrace{\left(|\vec{B}|^2 - \frac{3}{2}|\vec{B}|^2\right)}_{\rightarrow\ \frac{1}{8\pi}|\vec{B}|^2}$$

$$= 2\,\mathcal{E}_k + 3(\gamma-1)\,\mathcal{E}_{th} + \mathcal{E}_m$$

After integrating over volume, and combining with gravity term:

$$\boxed{2E_k + 3(\gamma-1)E_{th} + E_m + E_G = \frac{1}{2}\ddot{I} + \frac{1}{2}d_t\int r^2\rho\vec{u}\cdot\hat{n}\,dA + \oint \vec{x}\cdot(\rho\vec{u}\vec{u})\cdot\hat{n}\,dA + \oint P\,\vec{x}\cdot\hat{n}\,dA + \oint \vec{x}\cdot\left(\frac{I\frac{B^2}{2} - \vec{B}\vec{B}}{4\pi}\right)\cdot\hat{n}\,dA}$$

- Time-steady system or statistical steady state for ensemble $\Rightarrow \ddot{I}=0$, $d_t[\,]=0$
- Surface terms not necessarily negligible

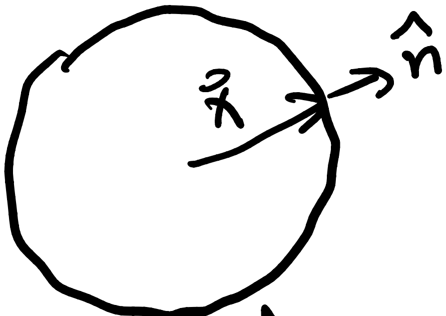
If surface terms negligible and $\dot{I} = 0$

$$2E_k + 3(\gamma-1)E_{th} + E_m + E_G = 0$$

$E_{th}, E_m = 0 \Rightarrow 2E_k + E_G = 0$ (mechanics)

$E_k, E_m = 0 \Rightarrow 3(\gamma-1)E_{th} + E_G = 0$ (stellar interiors)

What about surface terms?



spherical surface

$\hat{n} = \hat{r}$ in spherical coords

$\vec{x} = \vec{r}$

Reynolds: $\oint \rho u_r^2 \, r \, r^2 d\mu d\varphi$

Thermal: $\oint P \quad r \, r^2 d\mu d\varphi$

Maxwell: $\oint \frac{1}{4\pi} \left(\frac{B^2}{2} - B_r^2 \right) r \, r^2 d\mu d\varphi$

IF $\vec{u}$, $\vec{B}$ either uniform or isotropic on surface,

$\langle u_r^2 \rangle \to \langle \frac{1}{3} |\vec{u}|^2 \rangle$

$\langle \frac{B^2}{2} - B_r^2 \rangle \to \frac{\langle \vec{B}^2 \rangle}{6}$

Reynolds: $2 \frac{1}{2} \langle \rho u^2 \rangle_{surface} \frac{4\pi}{3} r^3 = 2 \langle \mathcal{E}_k \rangle_{surf} \cdot (\text{vol.})$

Thermal: $3 \langle P \rangle_{surf.} \frac{4\pi}{3} r^3 = 3(\gamma-1) \langle \mathcal{E}_{th} \rangle_{surf} \cdot (\text{vol.})$

Maxwell: $\frac{\langle B^2 \rangle_{surf.}}{8\pi} \frac{4\pi}{3} r^3 = \langle \mathcal{E}_{mag} \rangle_{surf} \cdot (\text{vol.})$

$$\left[2 \langle \Delta \mathcal{E}_k \rangle_{vol} + 3(\gamma-1) \langle \Delta \mathcal{E}_{th} \rangle_{vol} + \langle \Delta \mathcal{E}_m \rangle_{vol} \right] (\text{volume})$$

$$= - E_{grav.}$$

where $\Delta \mathcal{E}_k = \mathcal{E}_k - \langle \mathcal{E}_k \rangle_{surf}$

etc.