1. **A mechanical Alfvén wave.** Suppose we have a perfectly conducting rectangular loop of height h and part of its width x immersed in a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ oriented out of the page. The loop has a mass m and inductance L. Ignore gravity.

(a) Give the loop an initial velocity $\boldsymbol{v} = v_0\hat{\boldsymbol{x}}$ to the right, so that the flux through the loop increases in time. What happens? Describe the motion in words.

(b) Solve for the motion analytically.

(c) Now suppose that the loop has some resistance R. How big should R be before resistance plays an appreciable role in the motion?

2. **Energy conservation in MHD.** In Prof. Kunz's lecture notes on hydrodynamics, an equation was derived for the evolution of the total energy density (see (II.20)):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + e + \rho\Phi\right) + \boldsymbol{\nabla\cdot}\left[\left(\frac{1}{2}\rho u^2 + \gamma e + \rho\Phi\right)\boldsymbol{u}\right] = \rho\frac{\partial\Phi}{\partial t}, \tag{1}$$

where $e = P/(\gamma-1)$, Φ is the gravitational potential, and the other symbols have their usual meanings. Following Prof. Brown's lecture on ideal MHD, which presented the ideal-MHD induction equation,

$$\frac{\partial\boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}), \tag{2}$$

generalize the conservation law (1) to account for the evolution of the magnetic energy density, $B^2/8\pi$. In particular, demonstrate (a) that the magnetic energy is transported by the Poynting flux $\boldsymbol{S} \doteq c\boldsymbol{E}\times\boldsymbol{B}/4\pi$, and (b) that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + e + \rho\Phi + \frac{B^2}{8\pi}\right) + \boldsymbol{\nabla\cdot}\left[\left(\frac{1}{2}\rho u^2 + \gamma e + \rho\Phi\right)\boldsymbol{u} + \boldsymbol{S}\right] = \rho\frac{\partial\Phi}{\partial t}. \tag{3}$$

3. **Transport of energy by a circularly polarized Alfvén wave.** A circularly polarized Alfvén wave of amplitude $\delta B_\perp$ propagates along an otherwise uniform magnetic field $B_0\hat{\boldsymbol{z}}$:

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta B_\perp\,\boldsymbol{e}_\perp(t,z) \quad \text{and} \quad \boldsymbol{u} = -\frac{\delta B_\perp}{\sqrt{4\pi\rho}}\,\boldsymbol{e}_\perp(t,z), \tag{4}$$

where

$$\boldsymbol{e}_\perp(t,z) = \cos[k(v_{\rm A}t - z)]\hat{\boldsymbol{x}} + \sin[k(v_{\rm A}t - z)]\hat{\boldsymbol{y}}. \tag{5}$$

(a) Draw the magnetic-field line at $t = 0$. Which way is the wave propagating? Is the wave right-handed or left-handed?

(b) Prove that the magnetic-field strength B is a constant, despite the presence of the wave.

(c) Show that (4) is an exact nonlinear solution of the ideal-MHD equations.

(d) Calculate the time-averaged Poynting flux $\langle \boldsymbol{S} \rangle_t \doteq \langle c\boldsymbol{E}\times\boldsymbol{B}/4\pi \rangle_t$ for this wave. Write it in terms of the total wave energy $\mathcal{E} = \rho u^2/2 + \delta B_\perp^2/8\pi$. Interpret your result physically.

4. **Energy conservation in MHD turbulence.** In this problem, you will derive a conservation law for an *incompressible* (i.e., $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$) turbulent fluid.

(a) The MHD induction equation including a constant magnetic resistivity η is

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \frac{c^2\eta}{4\pi}\nabla^2\boldsymbol{B}, \tag{6}$$

where $\mathrm{D}/\mathrm{D}t \doteq \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ is the Lagrangian derivative. Dot this equation with $\boldsymbol{B}$ and integrate over real space to derive an evolution equation for the magnetic energy $\int \mathrm{d}^3\boldsymbol{r}\,|\boldsymbol{B}|^2/8\pi$. Assume zero flux at the boundaries (taken to be at infinity). (You may use the answer from Problem #2 and then add on the appropriate resistive contribution.)

(b) The MHD momentum equation including a constant dynamical viscosity μ is

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}}{4\pi} + \mu\nabla^2\boldsymbol{u}. \tag{7}$$

Dot this equation with $\boldsymbol{u}$ and integrate over real space to derive an evolution equation for the kinetic energy $\int \mathrm{d}^3\boldsymbol{r}\,\rho|\boldsymbol{u}|^2/2$. Again, assume zero flux at the boundaries (taken to be at infinity). At some point you'll need the mass continuity equation for an incompressible fluid,

$$\frac{\partial\rho}{\partial t} = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\rho.$$

(Note: this is done in Prof. Kunz's hydrodynamics lecture notes for an inviscid fluid, so you need only figure out the appropriate viscous contribution.)

(c) Add the results of parts (a) and (b) together to obtain the following conservation law for the total mechanical (magnetic + kinetic) energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}\left(\frac{|\boldsymbol{B}|^2}{8\pi} + \frac{\rho|\boldsymbol{u}|^2}{2}\right) = -\int \mathrm{d}^3\boldsymbol{r}\left(\eta|\boldsymbol{j}|^2 + \mu|\boldsymbol{\omega}|^2\right) \le 0, \tag{8}$$

where $\boldsymbol{j} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$ is the current density and $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ is the flow vorticity. Suppose there were a source term injecting mechanical energy into the system at large scales. Explain physically how a steady state ($\mathrm{d}/\mathrm{d}t = 0$) might be achieved. (Hint: what if η and μ were really, really small – would it matter?)

5. **Kelvin's circulation theorem** is an extremely important result in fluid dynamics. Every time you ride an airplane, you owe your life to it. In this problem you will prove it, as well as investigate the effects of baroclinicity and the Lorentz force.

(a) Start by taking the curl of the MHD force equation

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla} P + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho},$$

where $\boldsymbol{j} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$ is the current density, to obtain an evolution equation for the *vorticity* $\boldsymbol{\omega} \doteq \boldsymbol{\nabla}\times\boldsymbol{u}$. Use a particular vector identity to write it in the form $\partial\boldsymbol{\omega}/\partial t = \boldsymbol{\nabla}\times(\dots)$. This should look *almost* like the ideal-MHD induction equation, but not quite. (The resolution of this "not quite" involves freezing the magnetic field in the electron fluid and retaining a non-ideal term in the induction equation that allows the magnetic field to drift through the ion species – the so-called "Hall effect".)

Note: The vorticity is divergence free, which means that vortex lines cannot end within the fluid – they must either close on themselves (like a smoke ring) or intersect a boundary (like a tornado). Any fresh vortex lines that are made must be created as continuous curves that grow out of points or lines were the vorticity vanishes.

(b) The *circulation* is defined by

$$\Gamma \doteq \oint_{\partial\mathcal{S}} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{\ell}, \tag{9}$$

where $\mathrm{d}\boldsymbol{\ell}$ is an infinitesimal line element along a simple closed contour $\partial\mathcal{S}(t)$ bounding a material surface $\mathcal{S}(t)$ moving with velocity $\boldsymbol{u}$. By Stokes' theorem, this is equivalent to

$$\Gamma = \int_{\mathcal{S}} \boldsymbol{\omega}\cdot \mathrm{d}\boldsymbol{\mathcal{S}},$$

which states that the circulation around the boundary $\partial\mathcal{S}$ can be calculated as the number of vortex lines that thread the enclosed area $\mathcal{S}$. Take $\mathrm{D}/\mathrm{D}t$ of (9) and use the result of part (a) to obtain an equation for the evolution of the circulation. (Hint: don't forget to compute the time rate of change of the area, something that was done in the MHD lecture when proving Alfvén's theorem.)

Note: "simple closed contour" means *simply connected* – that is, the region must be such that we can shrink the contour to a point without leaving the region. A region with a hole (like a bathtub drain) is *not* simply connected.

(c) Answer all of the following questions. Under what conditions is the circulation conserved? Why would pressure gradients and density gradients have anything to do with generating circulation? Draw a picture as part of your explanation. (Hint: It might help to imagine what would happen in an atmosphere if the pressure gradient were being imposed by vertical gravity and the density of air were greater in the east than the west.) Why would the Lorentz force have anything to do with generating circulation? (Hint: Take an irrotational fluid and thread it with a twisted magnetic field. Let it go. What would happen?) Would it help or hurt vorticity conservation if the magnetic field weren't perfectly frozen into the plasma? Why?

(d) Knowing that the Coriolis force is $-2\boldsymbol{\Omega}\times\boldsymbol{u}$, prove that the circulation in a rotating reference frame is given by $\Gamma + \int_{\mathcal{S}} 2\boldsymbol{\Omega}\cdot\mathrm{d}\boldsymbol{\mathcal{S}}$, where $\boldsymbol{\Omega}$ is the angular velocity.

6. **Magnetorotational Instability with springs**. The acknowledgement at the end of Balbus & Hawley (1992a) reads, "It is fitting and proper to acknowledge Alar Toomre for this important insight that the Hill equations had something to contribute to the MHD stability problem." This insight is what led Balbus and Hawley to develop the now-famous spring model of the MRI, which was then used to conjecture that the Oort A-value is the universal growth rate limit for accretion-disk shear instabilities. The Hill equations describe local disk dynamics in a rotating frame – *local* in that they describe small excursions $x \doteq R - R_0$ and $y \doteq R_0(\varphi - \Omega_0 t)$ from a circular orbit $R = R_0$, $\varphi = \Omega_0 t$. They are given by:

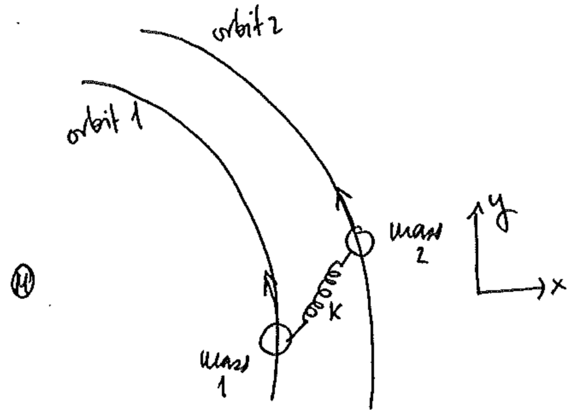
$$\ddot{x} - 2\Omega_0 \dot{y} = -4A_0\Omega_0 x + f_x, \tag{10a}$$
$$\ddot{y} + 2\Omega_0 \dot{x} = f_y, \tag{10b}$$

where the overdot indicates a time derivative and f_x and f_y represent local forces in the x and y directions. The Oort A-value $A_0 = -(3/4)\Omega_0$ for Keplerian rotation.[1]

The MRI analogy goes as follows. Consider the local force to be nondissipative and to act by restoring a displacement back to its equilibrium position. The leading-order contribution to f_x and f_y in a Taylor expansion about $(R_0, \Omega_0 t)$ is linear; for an *isotropic* force, we have $f_x = -Kx$ and $f_y = -Ky$, where $K > 0$ is some constant. (You could also profitably think of this force as being due to an ideal spring with spring constant K.) Then (10) becomes

$$\ddot{x} - 2\Omega_0 \dot{y} = -4A_0\Omega_0 x - Kx, \tag{11a}$$
$$\ddot{y} + 2\Omega_0 \dot{x} = -Ky. \tag{11b}$$

Visually,



Now then. . .

(a) For small displacements x, y, show that the solutions are $\propto \exp(\pm \mathrm{i}\omega t)$ with

$$\omega^4 - \omega^2\big(\kappa^2 + 2K\big) + K\big(K + 4A_0\Omega_0\big) = 0, \tag{12}$$

where $\kappa^2 \doteq 4\Omega_0^2(1 + A_0/\Omega_0)$ is the square of the epicyclic frequency, which is positive for Keplerian rotation. Equation (12) should look familiar from the lecture notes on MHD instabilities: set $K = 0$ and you get trivial displacements ($\omega^2 = 0$) and epicycles ($\omega^2 = \kappa^2$); replace K with $(\boldsymbol{k} \cdot \boldsymbol{v}_\mathrm{A})^2$ and you get the axisymmetric MRI linear dispersion relation. Show that $A_0 < 0$ is a necessary (but not sufficient) condition for instability.

[1] The notation for differential rotation varies in the accretion-disk literature; here's a dictionary: $2A_0 = -q\Omega_0 = \mathrm{d}\Omega/\mathrm{d}\ln R|_{R=R_0}$. Often, the "0" subscript is simply dropped for ease of notation.

(b) S. A. Balbus and J. F. Hawley, *Astrophys. J.* **392**, 662 (1992) conjecture "that the Oort A-value is an upper bound to the growth rate of any instability feeding upon the free energy of differential rotation." En route, they show that the maximum growth rate of the MRI is the Oort-A value, that it occurs at $K_{\max}/\Omega_0^2 = -(A_0/\Omega_0)(2 + A_0/\Omega_0)$, and that the corresponding eigenvector satisfies $y/x = -1$, i.e., radial and azimuthal displacements are equal in size. Prove these three facts.

(d) **Bonus**. Set $f_x = -K_x x$ and $f_y = -K_y y$ with $K_x \neq K_y$ being positive constants. Compute the new dispersion relation governing the time-evolution of small displacements. Is the growth rate larger or smaller than the Oort-A value for $K_x > K_y$? for $K_x < K_y$? From this result, find the maximum growth rate $\gamma_{\max}$ and the (hint: asymptotic) values of K_x and K_y at which $\gamma_{\max}$ is achieved. (It may help to make a quick contour plot of the growth rate in the K_x–K_y plane using your dispersion relation.) E. Quataert, W. Dorland, and G. W. Hammett *Astrophys. J.* **577**, 524 (2002) used this as a model for the magnetorotational instability in a collisionless plasma.

7. **Drifts in Dipoles**. The equation for a dipole magnetic field in spherical coordinates is given by

$$\boldsymbol{B} = \frac{3\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{r})}{r^5} - \frac{\boldsymbol{m}}{r^3} = \frac{m}{r^3}\left(2\cos\vartheta\,\hat{\boldsymbol{r}} + \sin\vartheta\,\hat{\boldsymbol{\vartheta}}\right), \tag{13}$$

where $\boldsymbol{m} = m\hat{\boldsymbol{z}}$ is the magnetic moment.

(a) Show that the equation for a magnetic-field line is $r = R\sin^2\vartheta$, where R is the radius of the magnetic-field line at the equator ($\vartheta = \pi/2$).

(b) Show that the curvature of the magnetic-field line at the equator ($\vartheta = \pi/2$) is $R_{\rm c} = R/3$.

(c) Compute the curvature drift of a particle with charge q and parallel kinetic energy $W_\parallel$ at a radial distance R at the equator.

(d) Compute the grad-B drift of a particle with charge q and perpendicular kinetic energy $W_\perp$ at a radial distance R at the equator. For what ratio $W_\perp/W_\parallel$ are the drifts the same?

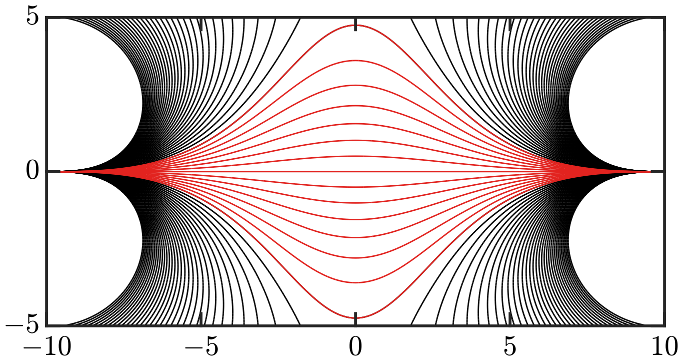
Now suppose there are two aligned magnetic dipoles with moment $\boldsymbol{m}$ spatially separated by $2\boldsymbol{a}$ about the origin. The magnetic field is then given by

$$\boldsymbol{B}(\boldsymbol{r}) = \left[\frac{3\boldsymbol{r}_+(\boldsymbol{m}\cdot\boldsymbol{r}_+)}{r_+^5} - \frac{\boldsymbol{m}}{r_+^3}\right] + \left[\frac{3\boldsymbol{r}_-(\boldsymbol{m}\cdot\boldsymbol{r}_-)}{r_-^5} - \frac{\boldsymbol{m}}{r_-^3}\right], \tag{14}$$

where $\boldsymbol{r}_\pm \doteq \boldsymbol{r} \pm \boldsymbol{a}$. This field may be obtained by taking the curl of the vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\boldsymbol{m}\times\boldsymbol{r}_+}{r_+^3} + \frac{\boldsymbol{m}\times\boldsymbol{r}_-}{r_-^3}. \tag{15}$$

Because $\partial\boldsymbol{A}/\partial t = \boldsymbol{0}$, we have $\boldsymbol{E} = \boldsymbol{0}$. Some magnetic-field lines in the $z = 0$ plane, obtained from the isocontours of A_y, are shown below, with those in red revealing a magnetic bottle:

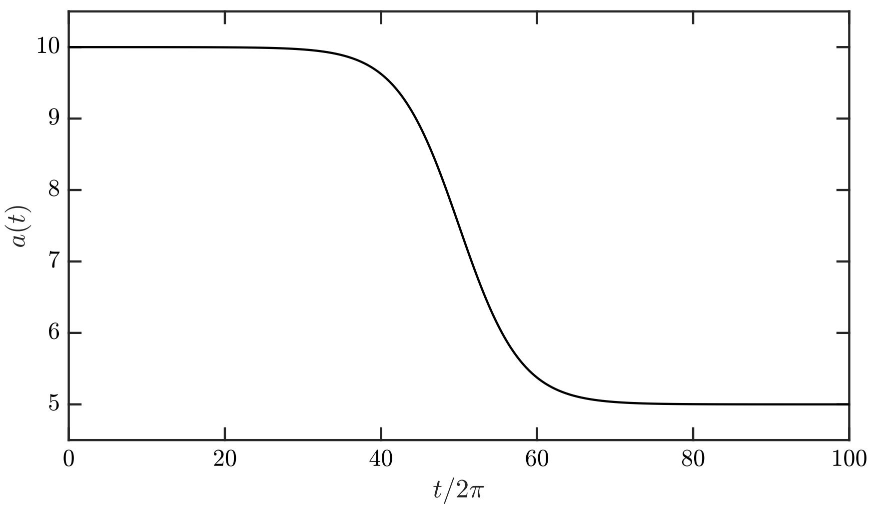


(e) Place a particle in the center of the mirror and launch it with velocity $\boldsymbol{v}$. Discuss with your group how the particle moves for various initial pitch angles, $v_x(0)/v(0)$.

(f) Suppose the distance between the two dipoles in part (e) is adiabatically shrunk in half:

$$\boldsymbol{a} \rightarrow \boldsymbol{a}(t) = 10\hat{\boldsymbol{x}} - 2.5\big\{1 + \tanh[\gamma(t - t_{\mathrm{f}}/2)]\big\}\hat{\boldsymbol{x}},$$

with $\gamma \ll 1$, as shown in the figure below:



The vector potential defined by equation (15) then depends upon time, $\boldsymbol{A}(\boldsymbol{r}) \rightarrow \boldsymbol{A}(t, \boldsymbol{r})$, and so there is a non-zero electric field, $\boldsymbol{E}(t, \boldsymbol{r}) = -\partial \boldsymbol{A}/\partial t$. Discuss with your group how the particle will move if $\boldsymbol{v}(0) = (\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})/\sqrt{2}$ (i.e., an initial pitch angle of 45°). In particular, what will $v_\parallel = \boldsymbol{v} \cdot \hat{\boldsymbol{b}}$ look like versus time?

8. **Critical balance.** In a rigidly rotating, hydrodynamic, incompressible fluid, the characteristic linear frequency of waves is $\omega = \pm(k_\parallel/k)\Omega$, where $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$ is the angular velocity of the flow and $k_\parallel = k_z$ is component the wavenumber oriented parallel to the rotation axis. Suppose that such a fluid is turbulent, with velocity fluctuations satisfying $k_\parallel/k_\perp \ll 1$, i.e., the fluctuations are anisotropic with respect to the rotation axis and elongated in that direction. Assume the turbulence to be strong and critically balanced. Obtain the resulting perpendicular and parallel power spectra of the turbulent velocities and the scaling relation linking $k_\parallel$ and $k_\perp$. Does the anisotropy of the fluctuations increase or decrease as the cascade goes to smaller scales? Is the similar to or different than Goldreich–Sridhar turbulence?

9. **Landau damping via Newton's 2nd.** Imagine an electron moving along the z axis with constant speed v_0. Slowly turn on a wave-like electric field: $\boldsymbol{E}(t,z) = E_0 \cos(\omega t - kz)\,\mathrm{e}^{\epsilon t}\hat{\boldsymbol{z}}$, where ω is the frequency and k is the wavenumber of the wave; the adverb "slowly" is captured by the $\mathrm{e}^{\epsilon t}$ factor with $\epsilon \ll 1$. (You'll take $\epsilon \to +0$ at the end of the calculation.) The goal is to solve this problem perturbatively by assuming E_0 is so small that it changes the electron's trajectory only a little bit over several wave periods.

(a) The lowest-order solution is $v_z(t) = v_0$, $z(t) = v_0 t$, and $E(t, v_0 t) = E_0 \cos[(\omega - kv_0)t]\,\mathrm{e}^{\epsilon t}$. Calculate the first-order corrections, $\delta v_z(t)$, $\delta z(t)$, and $\delta E(t,z)$.

(b) The average power gained by the electron (and thus lost by the wave) is

$$P(v_0) = -e\langle E(t, z(t)) v_z(t)\rangle \approx -e\langle [E(t, v_0 t) + \delta E(t,z)][v_0 + \delta v_z(t)]\rangle,$$

where the brackets indicate an average over timescales satisfying $\omega^{-1} \ll t \ll \epsilon^{-1}$. Use this to show that, to leading order,

$$P(v_0) = \frac{e^2 E_0^2}{2m_\mathrm{e}}\,\mathrm{e}^{2\epsilon t}\,\frac{\mathrm{d}\chi}{\mathrm{d}v_0}, \quad \text{where} \quad \chi(v_0) \doteq \frac{\epsilon v_0}{(\omega - kv_0)^2 + \epsilon^2}. \tag{16}$$

Plot $\chi(v_0)$ and identify when $P(v_0) > 0$ and $P(v_0) < 0$. Explain what each case means physically.

(c) This must be a very lonely electron, so let's give him some friends. Suppose there is now a whole distribution of these electrons, $F(v_0)$. Show that the total power per unit volume going into (or out of) this distribution is (take $\epsilon \to +0$)

$$P = -\frac{e^2 E_0^2}{2m_\mathrm{e}k^2}\pi\omega F'\left(\frac{\omega}{k}\right). \tag{17}$$

Explain this formula in the context of Landau damping. You'll need Plemelj's formula:

$$\lim_{\epsilon\to+0} \frac{1}{x - \zeta \mp \mathrm{i}\epsilon} = \mathrm{PV}\frac{1}{x-\zeta} \pm \mathrm{i}\pi\delta(x-\zeta),$$

where PV denotes the principal value and $\delta(x)$ is the Dirac delta function.