Note: There is much more here than you could possibly do during this school, no matter your background. But you can take these problems with you and learn from them over time. To guide you, each problem is given a ski-slope rating according to its intended difficulty: ●, ■, ◆.

Hydrodynamics

1. ● **Shrinking sink streams.** Go to the bathroom and turn on the sink slowly to get a nice, laminar stream flowing down from the faucet. Go on, I'll wait. If you followed instructions, then you'll see that the stream becomes more narrow as it descends. Knowing that the density of water is very nearly constant, use the continuity equation to show that the cross-sectional area of the stream $A(z)$ as a function of distance from the faucet z is

$$A(z) = \frac{A_0}{\sqrt{1 + 2gz/v_0^2}},$$

where A_0 is the cross-sectional area of the stream upon exiting the faucet with velocity v_0 and g is the gravitational acceleration. If you turn the faucet to make the water flow faster, what happens to the tapering of the stream?

2. ● **Self-gravity is stressful.** Show that the gravitational force on a self-gravitating fluid element may be written as

$$-\rho \boldsymbol{\nabla} \Phi = -\boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{g}\boldsymbol{g}}{4\pi G} - \frac{g^2}{8\pi G} \mathsf{\boldsymbol{I}} \right),$$

where $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, $g^2 = \boldsymbol{g} \cdot \boldsymbol{g}$, $\mathsf{\boldsymbol{I}}$ is the unit dyadic, and G is Newton's gravitational constant. The quantity inside the divergence operator is known as the gravitational stress tensor. Written in the form of a divergence, the gravitational force represents the flux of total momentum through a surface due to gravitational forces.

3. ● **Straining in cylindricals.** Show that the $R\varphi$-component in cylindrical coordinates of the rate-of-strain tensor

$$W_{ij} \doteq \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k}$$

is given by

$$W_{R\varphi} = \frac{1}{R} \frac{\partial u_R}{\partial \varphi} + R \frac{\partial}{\partial R} \frac{u_\varphi}{R}.$$

(Hint: $\partial u_i / \partial x_j = [(\hat{\boldsymbol{e}}_j \cdot \boldsymbol{\nabla})\boldsymbol{u}] \cdot \hat{\boldsymbol{e}}_i$ is coordinate invariant.) Such a combination often shows up in the theory of angular-momentum transport in accretion discs.

4. ■ **Helicity conservation.** Given the vorticity $\boldsymbol{\omega} \doteq \boldsymbol{\nabla} \times \boldsymbol{u}$, the helicity of a region of fluid is defined to be $\mathcal{H} \doteq \int \mathrm{d}\mathcal{V}\, \boldsymbol{\omega} \cdot \boldsymbol{u}$, where the integral is taken over the volume of that region. Assume that the circulation $\Gamma = \text{const}$ and that $\boldsymbol{\omega} \cdot \hat{\boldsymbol{n}}$ vanishes over the surface bounding $\mathcal{V}$,

where $\hat{\boldsymbol{n}}$ is the unit normal to that surface. Prove that the helicity $\mathcal{H}$ is conserved in a frame moving with the fluid, *viz.* $\mathrm{D}\mathcal{H}/\mathrm{D}t = 0$. Note that the fluid need not be incompressible for this property to hold.

5. ◆ **Spiral density waves and inertial waves.** Section IV.4 of Kunz's lecture notes contains a linear analysis of an unmagnetized, adiabatic, self-gravitating fluid. With P and ρ being the background thermal pressure and mass density (both taken to be uniform), the dispersion relation governing small-amplitude perturbations was $\omega^2 - k^2a^2 + 4\pi G\rho = 0$ with $a^2 \doteq \gamma P/\rho$. Solutions were "Jeans unstable" for $k^2a^2 < 4\pi G\rho$. This problem has you repeat this linear analysis, but in cylindrical coordinates (R, φ, z) for a differentially rotating disk with angular frequency $\boldsymbol{\Omega} = \Omega(R)\hat{\boldsymbol{z}}$. Your starting point will be §II.5 of Kunz's lecture notes, where you will find the hydrodynamic equations written in a rotating frame. In what follows, take the background pressure to be barotropic and allow the background $\rho = \rho(R, z)$.

(a) Take the perturbations to have space-time dependence $\exp(-\mathrm{i}\omega t + \mathrm{i}m\varphi + \mathrm{i}k_R R + \mathrm{i}k_z z)$ with $k_R L_R \sim k_R R \gg 1$ and $k_z L_z \gg 1$, where L_R (L_z) is the characteristic disk lengthscale in the radial (vertical) direction. (This is a WKB approximation: the perturbations are assumed to vary on lengthscales much shorter than those characterizing the background.) Obtain the following dispersion relation in the "tightly wound" limit in which both k_R and $k_z \gg m/R$:

$$\varpi^4 - \varpi^2\big(\kappa^2 + k^2a^2 - 4\pi G\rho\big) + \kappa^2\frac{k_z^2}{k^2}\big(k^2a^2 - 4\pi G\rho\big) = 0,$$

where $\varpi \doteq \omega - m\Omega$ is a Doppler-shifted frequency, $\kappa^2 = 4\Omega^2 + \mathrm{d}\Omega^2/\mathrm{d}\ln R$ is the square of the epicyclic frequency, and $k^2 = k_R^2 + k_z^2$. Another way to write this result is

$$\varpi^2 - k^2a^2 + 4\pi G\rho = \frac{\kappa^2 k_R^2 a^2}{\varpi^2 - \kappa^2}\left(1 - \frac{4\pi G\rho}{k^2a^2}\right),$$

which has the usual Jeans dispersion relation on the left-hand side (but for $\omega^2 \to \varpi^2$) and has a right-hand side that includes effects associated with the differential rotation.

(b) Consider the case $k_z = 0$. The result is the dispersion relation for *spiral density waves*:

$$\varpi^2 = \kappa^2 + k^2a^2 - 4\pi G\rho.$$

Such waves are thought to be particularly important in theories of galactic structure and protostellar disks. Note that rotation is a stabilizing influence (as is differential rotation if $\kappa^2 > 0$, the usual situation in astrophysical disks). Physically, why?

(c) Now take $k_z a \gg \kappa$ and $(4\pi G\rho)^{1/2}$ to obtain the dispersion relation for *inertial waves*:

$$\varpi^2 = \frac{k_z^2}{k^2}\kappa^2.$$

These waves are essentially incompressible, and are the only fluctuations in a polytropic, non-self-gravitating disk with frequencies less than κ. Note the dependence on k_z, which in concert with their incompressible nature tells us that the fluid displacements in this wave are primarily in the disk plane. With that in mind, what force is responsible for this wave?

(d) ◆◆ Repeat the calculation in part (a) but *without* adopting the WKB approximation. Namely, take the perturbations to have the form $f(R,z)\exp(-\mathrm{i}\omega t+\mathrm{i}m\varphi)$ and obtain the following linear wave equation for the potential $\delta h \doteq \delta P/\rho+\delta\Phi$:

$$\left[\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{R\rho}{D}\frac{\partial}{\partial R}\right)-\frac{1}{\varpi^2}\frac{\partial}{\partial z}\left(\rho\frac{\partial}{\partial z}\right)-\frac{m^2\rho}{R^2D}+\frac{1}{\varpi R}\frac{\partial}{\partial R}\left(\frac{2\Omega m\rho}{D}\right)\right]\delta h=-\delta\rho,$$

where $D \doteq \kappa^2-\varpi^2$ and $\delta\Phi$ is the solution to the linearized Poisson's equation. Note the resonances at $D=0$ and $\varpi=0$, which are referred to as the Lindblad and corotation resonances, respectively. Near these resonances, the waves couple strongly to the disk. (The WKB treatment formally breaks down at the Lindblad resonance, at which k_R must vanish.) These resonances are important in the study of tidally driven waves and planetary migration. For more on this topic, see Goldreich & Tremaine (1979, *Astrophys. J.* **233**, 857) and Balbus (2003, *Annu. Rev. Astron. Astrophys.* **41**, 555).

Magnetohydrodynamics: Waves

6. ● **A mechanical Alfvén wave.** Suppose we have a perfectly conducting rectangular loop of height h and part of its width x immersed in a uniform magnetic field $\boldsymbol{B}=B\hat{\boldsymbol{z}}$ oriented out of the page. The loop has a mass m and inductance L. Ignore gravity.

(a) Give the loop an initial velocity $\boldsymbol{v}=v_0\hat{\boldsymbol{x}}$ to the right, so that the flux through the loop increases in time. What happens? Describe the motion in words.

(b) Solve for the motion analytically.

(c) Now suppose that the loop has some resistance R. How big should R be before resistance plays an appreciable role in the motion?

7. ● **Transport of energy by an Alfvén wave.** A circularly polarized Alfvén wave of amplitude $\delta B_\perp$ propagates along an otherwise uniform magnetic field $B_0\hat{\boldsymbol{z}}$:

$$\boldsymbol{B}=B_0\hat{\boldsymbol{z}}+\delta B_\perp\,\boldsymbol{e}_\perp(t,z) \quad\text{and}\quad \boldsymbol{u}=-\frac{\delta B_\perp}{\sqrt{4\pi\rho}}\,\boldsymbol{e}_\perp(t,z), \tag{1}$$

where

$$\boldsymbol{e}_\perp(t,z)=\cos[k(v_\mathrm{A}t-z)]\hat{\boldsymbol{x}}+\sin[k(v_\mathrm{A}t-z)]\hat{\boldsymbol{y}}.$$

(a) Draw the magnetic-field line at $t=0$. Which way is the wave propagating?

(b) Prove that the magnetic-field strength B is a constant, despite the presence of the wave.

(c) Show that (1) is an exact nonlinear solution of the ideal-MHD equations.

(d) Calculate the time-averaged Poynting flux $\langle\boldsymbol{S}\rangle_t \doteq \langle c\boldsymbol{E}\times\boldsymbol{B}/4\pi\rangle_t$ for this wave. Write it in terms of the total wave energy $\mathcal{E}=\rho u^2/2+\delta B_\perp^2/8\pi$. Interpret your result physically.

Magnetohydrodynamics: Conservation laws

8. ● **Kelvin's circulation theorem in MHD**. In §II.4 of Kunz's lecture notes, Kelvin's circulation theorem was proven for the case without a magnetic field. Here you will generalize it for MHD. First, a reminder of the hydrodynamic case:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} \doteq \frac{\mathrm{D}}{\mathrm{D}t}\int_{\partial\mathcal{S}} \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{\ell} = \frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{S}} \boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{\mathcal{S}} = \oint_{\partial\mathcal{S}}\left(-\frac{1}{\rho}\boldsymbol{\nabla}P\right)\cdot\mathrm{d}\boldsymbol{\ell} = -\oint_{\partial\mathcal{S}}\frac{\mathrm{d}P}{\rho},$$

where $\boldsymbol{\omega} \doteq \boldsymbol{\nabla}\times\boldsymbol{u}$ is the vorticity. For a barotropic fluid, $\Gamma = \text{const}$ in the frame of the flow.

(a) Use the MHD force equation to show that Kelvin's circulation theorem in MHD becomes

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \oint_{\partial\mathcal{S}}\left(-\frac{\mathrm{d}P}{\rho} + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho}\right)\cdot\mathrm{d}\boldsymbol{\ell},$$

where $\boldsymbol{j} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$.

(b) Explain how the Lorentz force could generate circulation. (Hint: Take an irrotational fluid and thread it with a twisted magnetic field. Let it go. What would happen?) Would it help or hurt vorticity conservation if the magnetic field weren't perfectly frozen into the plasma? Why?

9. ■ **Energy conservation in MHD.** In §II.1.3 of Kunz's lecture notes on hydrodynamics, an equation was derived for the evolution of the total energy density (see (II.20)):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + e + \rho\Phi\right) + \boldsymbol{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2 + \gamma e + \rho\Phi\right)\boldsymbol{u}\right] = \rho\frac{\partial\Phi}{\partial t}, \qquad (2)$$

where $e = P/(\gamma-1)$, Φ is the gravitational potential, and the other symbols have their usual meanings. Following Prof. Brown's lecture on ideal MHD, which presented the ideal-MHD induction equation,

$$\frac{\partial\boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}),$$

generalize the conservation law (2) to account for the evolution of the magnetic energy density, $B^2/8\pi$. In particular, demonstrate (a) that the magnetic energy is transported by the Poynting flux $\boldsymbol{S} \doteq c\boldsymbol{E}\times\boldsymbol{B}/4\pi$, and (b) that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + e + \rho\Phi + \frac{B^2}{8\pi}\right) + \boldsymbol{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2 + \gamma e + \rho\Phi\right)\boldsymbol{u} + \boldsymbol{S}\right] = \rho\frac{\partial\Phi}{\partial t}.$$

10. ■ **Lundquist's theorem.** The concept of flux freezing is usually introduced by way of Alfvén's theorem: the magnetic flux passing through a surface moving along with the fluid is conserved. There is an alternative description of flux freezing stated in terms of line tying: fluid elements that lie on a field line initially will remain on that field line (S. Lundquist, *Phys. Rev.* **83**, 2 (1951)). Starting from the ideal induction equation, $\partial\boldsymbol{B}/\partial t = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})$, use the continuity equation to show that

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\boldsymbol{B}}{\rho} = \frac{\boldsymbol{B}}{\rho}\cdot\boldsymbol{\nabla}\boldsymbol{u},$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$. By comparing this equation to that describing the evolution of an infinitesimal Lagrangian separation vector between two points in a moving fluid, argue that the magnetic field moves with the flow.

11. ◆ **Wöltjer–Taylor relaxation**. In some systems (e.g., the solar corona, experiments in plasma confinement using a toroidal pinch), the plasma evolves towards a preferred configuration known as the "relaxed state". This state is in a configuration of minimum magnetic energy, but a minimum energy subject to the constraint that the global magnetic helicity $H_0 \doteq \int_{\mathcal{V}_0} \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{A}\cdot\boldsymbol{B}$ is conserved. (Here, $\boldsymbol{A}$ is the vector potential satisfying $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ and $\mathcal{V}_0$ is the total volume of the isolated plasma under consideration). Helicity can be interpreted in a topological sense as the number of linkages of magnetic flux tubes with one another; you can read about this in pretty much any decent textbook on MHD. Even when the plasma is not ideal, helicity conservation seems to remain a fairly good approximation.[1]

(a) Show that, while $\boldsymbol{A}\cdot\boldsymbol{B}$ is not gauge invariant, its integral within a flux tube is. (Hint: recall from undergraduate electromagnetism that $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\psi$, where ψ is an arbitrary scalar function, changes nothing in Maxwell's equations.) State under what conditions H_0 is gauge invariant.

(b) Show that H_0 is a conserved quantity in ideal MHD (but not in resistive MHD).

(c) ◆◆ Use the variational principle to minimize magnetic energy subject to constant helicity:

$$\delta \int \mathrm{d}^3\boldsymbol{r}\, \big(B^2 - \alpha \boldsymbol{A}\cdot\boldsymbol{B}\big) = 0,$$

where α is the Lagrange multiplier introduced to enforce the constant-helicity constraint. Show that this procedure yields $\boldsymbol{\nabla}\times\boldsymbol{B} = \alpha\boldsymbol{B}$ (and thus $\nabla^2\boldsymbol{B} = -\alpha^2\boldsymbol{B}$, the Helmholtz equation), i.e., $\boldsymbol{B}$ is a linear force-free field. What boundary conditions must you impose to obtain this result? You may find it helpful to use flux freezing in the Lagrangian viewpoint, *viz.*, $\delta\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B})$, where $\boldsymbol{\xi}$ is the Lagrangian displacement of a fluid element (see §IV.3 of Kunz's lecture notes).

(d) Consider a relaxed (i.e., linear force-free) field with cylindrical symmetry: $\partial/\partial\varphi = 0$, $\partial/\partial z = 0$. Show that $B_z = B_0 J_0(\alpha R)$ and $B_\varphi = B_0 J_1(\alpha R)$, where J_n is the nth Bessel function and R is the cylindrical radius. This corresponds to a field twisted about a cylindrical surface ("cylindrical pinch").

If you're interested in learning more, consult J. B. Taylor (1986), RvMP, 58, 741.

[1] Some history: Wöltjer (1958) showed that there are an infinite number of integral invariants in ideal MHD: $H_i \doteq \int_{\mathcal{V}_i} \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{A}\cdot\boldsymbol{B} = \text{const}$ on each and every flux tube $\mathcal{V}_i$ in the system. These invariants are related to the well-known property that the magnetic field is frozen into an ideally conducting plasma. J. B. Taylor (1974) realized that, in a slightly resistive turbulent plasma contained within a perfectly conducting boundary, the only flux tube to retain its integrity is that which contains the entire plasma. Then, only H_0 will remain invariant. Taylor's conjecture is that MHD systems tend to minimize their magnetic energy subject to the constraint that the *total* magnetic helicity remains constant.

Magnetohydrodynamics: Instabilities

12. ◆ **Magnetorotational instability with springs**. The acknowledgement at the end of Balbus & Hawley (1992a) reads, "It is fitting and proper to acknowledge Alar Toomre for this important insight that the Hill equations had something to contribute to the MHD stability problem." This insight is what led Balbus and Hawley to develop the now-famous spring model of the MRI, which was then used to conjecture that the Oort A-value is the universal growth rate limit for accretion-disk shear instabilities. The Hill equations describe local disk dynamics in a rotating frame – *local* in that they describe small excursions $x \doteq R - R_0$ and $y \doteq R_0(\varphi - \Omega_0 t)$ from a circular orbit $R = R_0$, $\varphi = \Omega_0 t$. They are given by:

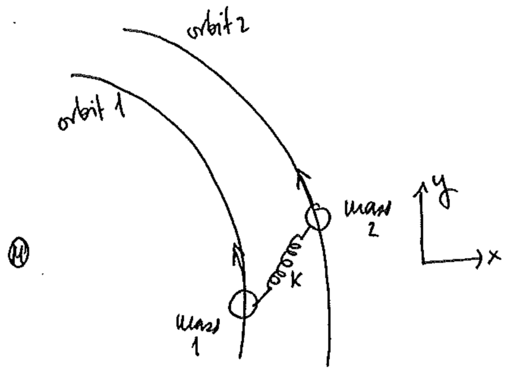
$$\ddot{x} - 2\Omega_0 \dot{y} = -4A_0\Omega_0 x + f_x, \tag{3a}$$
$$\ddot{y} + 2\Omega_0 \dot{x} = f_y, \tag{3b}$$

where the overdot indicates a time derivative and f_x and f_y represent local forces in the x and y directions. The Oort A-value $A_0 = -(3/4)\Omega_0$ for Keplerian rotation.[2]

The MRI analogy goes as follows. Consider the local force to be nondissipative and to act by restoring a displacement back to its equilibrium position. The leading-order contribution to f_x and f_y in a Taylor expansion about $(R_0, \Omega_0 t)$ is linear; for an *isotropic* force, we have $f_x = -Kx$ and $f_y = -Ky$, where $K > 0$ is some constant. (You could also profitably think of this force as being due to an ideal spring with spring constant K.) Then (3) becomes

$$\ddot{x} - 2\Omega_0 \dot{y} = -4A_0\Omega_0 x - Kx, \tag{4a}$$
$$\ddot{y} + 2\Omega_0 \dot{x} = -Ky. \tag{4b}$$

Visually,



Now then...

(a) For small displacements x, y, show that the solutions to (4) are $\propto \exp(\pm i\omega t)$ with

$$\omega^4 - \omega^2\big(\kappa^2 + 2K\big) + K\big(K + 4A_0\Omega_0\big) = 0, \tag{5}$$

where $\kappa^2 \doteq 4\Omega_0^2(1 + A_0/\Omega_0)$ is the square of the epicyclic frequency, which is positive for Keplerian rotation. Equation (5) should look familiar from the lecture notes on

[2] The notation for differential rotation varies in the accretion-disk literature; here's a dictionary: $2A_0 = -q\Omega_0 = (\mathrm{d}\Omega/\mathrm{d}\ln R)_{R=R_0}$. Often, the "0" subscript is simply dropped for ease of notation.

MHD instabilities: set $K = 0$ and you get trivial displacements ($\omega^2 = 0$) and epicycles ($\omega^2 = \kappa^2$); replace K with $(\boldsymbol{k} \cdot \boldsymbol{v}_{\rm A})^2$ and you get the axisymmetric MRI linear dispersion relation. Show that $A_0 < 0$ is a necessary (but not sufficient) condition for instability.

(b) S. A. Balbus and J. F. Hawley, *Astrophys. J.* **392**, 662 (1992) conjecture "that the Oort A-value is an upper bound to the growth rate of any instability feeding upon the free energy of differential rotation." En route, they show that the maximum growth rate of the MRI is the Oort-A value, that it occurs at $K_{\rm max}/\Omega_0^2 = -(A_0/\Omega_0)(2 + A_0/\Omega_0)$, and that the corresponding eigenvector satisfies $y/x = -1$, i.e., radial and azimuthal displacements are equal in size. Prove these three facts.

(c) Use these to show that, at maximum growth, the Lagrangian change in the rotation frequency of a displaced fluid element is $\Delta\Omega = \dot{y}/R_0 = -|A_0|x/R_0$ and that the corresponding Lagrangian change in its specific angular momentum $\ell = \Omega R^2$ satisfies

$$\frac{\Delta \ell}{\ell_0} = 2\frac{x}{R_0} + \frac{\Delta\Omega}{\Omega_0} = 2\left(1 - \frac{|A_0|}{2\Omega_0}\right)\frac{x}{R_0}.$$

Then show that outwardly (inwardly) displaced fluid elements always have more (less) angular momentum that the orbits they are passing through (which is what makes instability possible). (Hint: what is the difference in ℓ between two undisturbed orbits a radial distance x apart, in a disk in which $\mathrm{d}\Omega/\mathrm{d}\ln R = 2A_0 < 0$?)

(d) **Bonus**. Set $f_x = -K_x x$ and $f_y = -K_y y$ with $K_x \neq K_y$ being positive constants. Compute the new dispersion relation governing the time-evolution of small displacements. Is the growth rate larger or smaller than the Oort-A value for $K_x > K_y$? for $K_x < K_y$? From this result, find the maximum growth rate $\gamma_{\rm max}$ and the (hint: asymptotic) values of K_x and K_y at which $\gamma_{\rm max}$ is achieved. (It may help to make a quick contour plot of the growth rate in the K_x–K_y plane using your dispersion relation.) E. Quataert, W. Dorland, and G. W. Hammett *Astrophys. J.* **577**, 524 (2002) used this as a model for the magnetorotational instability in a collisionless, magnetized plasma.

Turbulence

13. ◆ **Critical balance.** In a rigidly rotating, hydrodynamic, incompressible fluid, the characteristic linear frequency of waves is $\omega = \pm(k_\parallel/k)\Omega$, where $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$ is the angular velocity of the flow and $k_\parallel = k_z$ is component the wavenumber oriented parallel to the rotation axis. (These are the "inertial waves" seen in Problem 5.) Suppose that such a fluid is turbulent, with velocity fluctuations satisfying $k_\parallel/k_\perp \ll 1$, i.e., the fluctuations are anisotropic with respect to the rotation axis and elongated in that direction. Assume the turbulence to be strong and critically balanced. Obtain the resulting perpendicular and parallel power spectra of the turbulent velocities and the scaling relation linking $k_\parallel$ and $k_\perp$. Does the anisotropy of the fluctuations increase or decrease as the cascade goes to smaller scales? Is the similar to or different than Goldreich–Sridhar turbulence?