Note: There is much more here than you could possibly do during this school, no matter your background. But you can take these problems with you and learn from them over time. To guide you, each problem is given a ski-slope rating according to its intended difficulty: ●, ■, ◆.

MHD shocks

1. ● **Earth's bow shock.** (Based on a problem from Thorne & Blandford) The Sun leaks $\sim 10^{-14}$ $\mathrm{M}_\odot$ yr^{-1} off its surface in the form of a supersonic, hydromagnetic flow of plasma. At the radius of the Earth's orbit (1 au), this "solar wind" is characterized by a bulk velocity $v \sim 400$ km s^{-1}, density $n \sim 10$ cm^{-3}, and temperature $T \sim 10^5$ K. It is threaded by an interplanetary magnetic field, arranged approximately in the shape of a spiral emerging from the Sun (as predicted by E. Parker), whose strength at 1 au is $B \sim 50$ μG.

(a) Balance the momentum flux of the solar wind with the magnetic pressure exerted by the Earth's dipolar magnetic field to estimate the radius above the Earth at which the solar wind passes through a bow shock. (Useful facts: The strength of the Earth's magnetic field at the surface is $B_\oplus \sim 0.5$ G. The radius of the Earth is $R_\oplus = 6371$ km.)

(b) Consider a strong perpendicular shock at which the magnetic field is parallel to the shock front. Show that the magnetic-field strength will increase by the same ratio as the density when crossing the shock front. Is there an upper limit to the factor by which a perpendicular shock can increase the magnetic field?

Magnetic reconnection

2. ■ **2D magnetic reconnection?** Consider a plasma that is rigorously described by the following resistive-MHD Ohm's law: $\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B} = \eta\boldsymbol{j}$. Suppose that the velocity and magnetic fields are two-dimensional, with $\boldsymbol{u} = u_x(t,x,y)\hat{\boldsymbol{x}} + u_y(t,x,y)\hat{\boldsymbol{y}}$ and $\boldsymbol{B} = B_x(t,x,y)\hat{\boldsymbol{x}} + B_y(t,x,y)\hat{\boldsymbol{y}}$, respectively. Use Ampère's law to show that $\boldsymbol{j} = j_z(t,x,y)\hat{\boldsymbol{z}}$. Then show that there is a velocity $\boldsymbol{v}$ such that $\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B} = 0$. Give an explicit expression for $\boldsymbol{v}$ in terms of $\boldsymbol{u}$, $\boldsymbol{j}$, and $\boldsymbol{B}$. Does this mean there is no reconnection in two dimensions? If not, why not?

Charged particle motion

3. ● **Drift currents.**

(a) Write down expressions for the E-cross-B drift $\boldsymbol{v}_E$, the grad-B drift $\boldsymbol{v}_{\nabla B}$, the curvature drift $\boldsymbol{v}_{\mathrm{c}}$, and the polarization drift $\boldsymbol{v}_{\mathrm{pol}}$.

(b) In an ion–electron plasma, which of these drifts have currents associated with them?

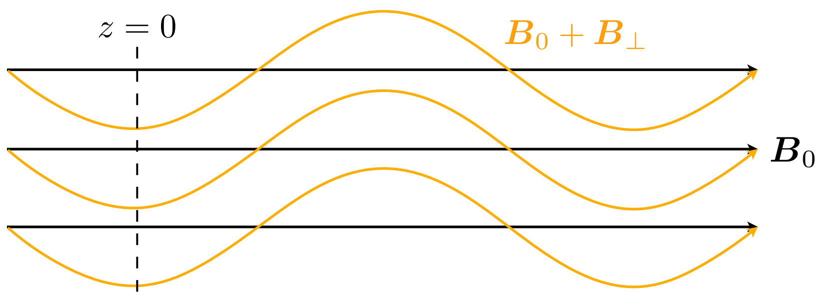
(c) Show that the current densities associated with the grad-B and curvature drifts are equal for a pressure-isotropic plasma in a force-free magnetic field having $\boldsymbol{j}\times\boldsymbol{B} = 0$. [Answer: $\boldsymbol{j}_{\nabla B} = \boldsymbol{j}_{\mathrm{c}} = (cP/B)\,\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\ln B$.]

4. **Drifts in MHD waves.**

(a) A small-amplitude linearly polarized Alfvén wave of amplitude $B_\perp$ and wavenumber $k > 0$ propagates along a uniform magnetic field $B_0\hat{\boldsymbol{z}}$ through an otherwise stationary, uniform, ideal-MHD plasma. The magnetic field and fluid velocity are given by

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + B_\perp \sin\big[k(z - v_\mathrm{A}t)\big]\hat{\boldsymbol{x}} \quad \text{and} \quad \boldsymbol{u} = -v_\mathrm{A}\frac{B_\perp}{B_0}\sin\big[k(z - v_\mathrm{A}t)\big]\hat{\boldsymbol{x}},$$

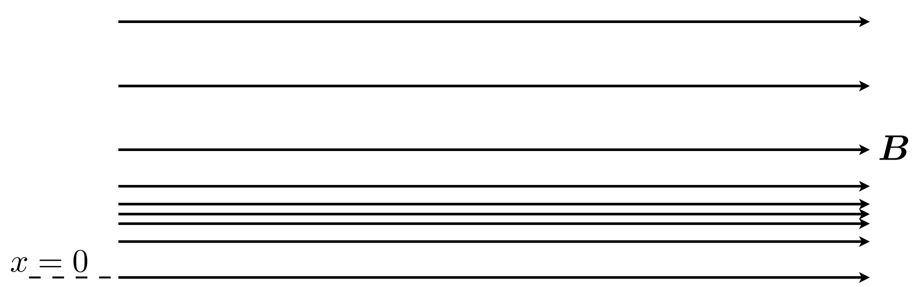
respectively, where $v_\mathrm{A} \doteq B_0/\sqrt{4\pi\rho}$ is the Alfvén speed. Neglecting terms of order $B_\perp^2$ and higher, compute all guiding-center drifts for this wave. Draw them for an electron on the figure below:



(b) A small-amplitude fast mode of amplitude $B_\parallel$ and wavenumber $k > 0$ propagates across a uniform magnetic field $B_0\hat{\boldsymbol{z}}$ through an otherwise stationary, uniform, ideal-MHD plasma. The magnetic field and fluid velocity are given by

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + B_\parallel \sin\big[k(x - v_\mathrm{f}t)\big]\hat{\boldsymbol{z}} \quad \text{and} \quad \boldsymbol{u} = v_\mathrm{f}\frac{B_\parallel}{B_0}\sin\big[k(x - v_\mathrm{f}t)\big]\hat{\boldsymbol{x}},$$

respectively, where $v_\mathrm{f} \doteq \sqrt{v_\mathrm{A}^2 + a^2}$ is the fast magnetosonic speed. Neglecting terms of order $B_\parallel^2$ and higher, compute all guiding-center drifts for this wave. Draw them for an electron on the figure below:



5. **Drifts in dipoles.** The equation for a dipole magnetic field in spherical coordinates is given by

$$\boldsymbol{B} = \frac{3\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{r})}{r^5} - \frac{\boldsymbol{m}}{r^3} = \frac{m}{r^3}\left(2\cos\vartheta\,\hat{\boldsymbol{r}} + \sin\vartheta\,\hat{\boldsymbol{\vartheta}}\right), \tag{1}$$

where $\boldsymbol{m} = m\hat{\boldsymbol{z}}$ is the magnetic moment.

(a) Show that the equation for a magnetic-field line is $r = R\sin^2\vartheta$, where R is the radius of the magnetic-field line at the equator ($\vartheta = \pi/2$).

(b) Show that the curvature of the magnetic-field line at the equator ($\vartheta = \pi/2$) is $R_\mathrm{c} = R/3$.

(c) Compute the curvature drift of a particle with charge q and parallel kinetic energy $W_\parallel$ at a radial distance R at the equator.

(d) Compute the grad-B drift of a particle with charge q and perpendicular kinetic energy $W_\perp$ at a radial distance R at the equator. For what ratio $W_\perp/W_\parallel$ are the drifts the same?

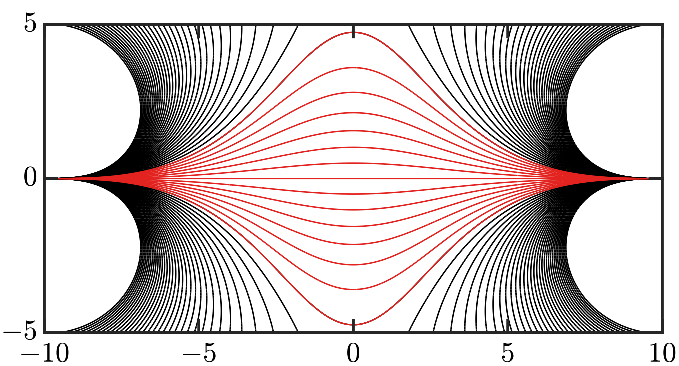
Now suppose there are two aligned magnetic dipoles with moment $\boldsymbol{m}$ spatially separated by $2\boldsymbol{a}$ about the origin. Using (1), the resulting magnetic field is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \left[\frac{3\boldsymbol{r}_+(\boldsymbol{m}\boldsymbol{\cdot}\boldsymbol{r}_+)}{r_+^5} - \frac{\boldsymbol{m}}{r_+^3}\right] + \left[\frac{3\boldsymbol{r}_-(\boldsymbol{m}\boldsymbol{\cdot}\boldsymbol{r}_-)}{r_-^5} - \frac{\boldsymbol{m}}{r_-^3}\right], \tag{2}$$

where $\boldsymbol{r}_\pm \doteq \boldsymbol{r} \pm \boldsymbol{a}$. This field may be obtained by taking the curl of the vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\boldsymbol{m}\times\boldsymbol{r}_+}{r_+^3} + \frac{\boldsymbol{m}\times\boldsymbol{r}_-}{r_-^3}. \tag{3}$$

Because $\partial\boldsymbol{A}/\partial t = \boldsymbol{0}$, we have $\boldsymbol{E} = \boldsymbol{0}$. Some magnetic-field lines in the $z = 0$ plane, obtained from the isocontours of A_y, are shown below, with those in red revealing a magnetic bottle:

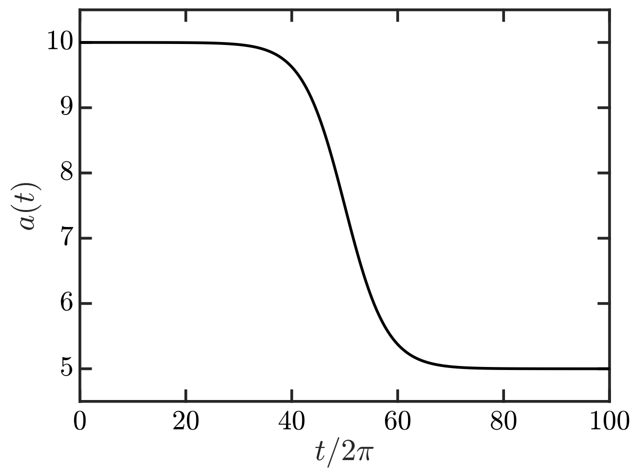


(e) Place a particle in the center of the mirror and launch it with velocity $\boldsymbol{v}$. Discuss with your group how the particle moves for various initial pitch angles, $v_x(0)/v(0)$.

(f) Suppose the distance between the two dipoles in part (e) is adiabatically shrunk in half:

$$\boldsymbol{a} \to \boldsymbol{a}(t) = 10\hat{\boldsymbol{x}} - 2.5\big\{1 + \tanh[\gamma(t - t_\mathrm{f}/2)]\big\}\hat{\boldsymbol{x}},$$

with $\gamma \ll 1$, as shown in the figure below:



The vector potential defined by equation (3) then depends upon time, $\boldsymbol{A}(\boldsymbol{r}) \to \boldsymbol{A}(t, \boldsymbol{r})$, and so there is a non-zero electric field, $\boldsymbol{E}(t, \boldsymbol{r}) = -\partial \boldsymbol{A}/\partial t$. Discuss with your group how the particle will move if $\boldsymbol{v}(0) = (\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})/\sqrt{2}$ (i.e., an initial pitch angle of 45°). In particular, what will $v_\parallel = \boldsymbol{v} \cdot \hat{\boldsymbol{b}}$ look like versus time?

Adiabatic invariance and pressure anisotropy

6. ◆ **Magnetic pumping.** Imagine a stationary, uniform, magnetically confined plasma whose thermal pressure P_0 is initially isotropic, i.e., $P_\perp(0) = P_\parallel(0) = P_0$, where $P_\perp$ is the perpendicular pressure and $P_\parallel$ is the parallel pressure. (Note that $3P = P_\parallel + 2P_\perp$.) Take the magnetic field to be uniform with initial strength B_0.

(a) Slowly increase the strength of this field from B_0 to B_1. "Slowly" here means that the rate of increase is slow compared with the gyro-frequency of the trapped particles but fast compared with the rate at which collisions establish isotropy of the particle distribution function. Use adiabatic invariance to argue that $P_\perp = (B_1/B_0)P_0$ and $P_\parallel = P_0$. Now wait long enough for energy-conserving Coulomb collisions to isotropize the temperature, so the $P_\perp = P_\parallel = P_1$. What is the value of P_1?

(b) Once the system is well equilibrated, decrease the magnetic-field strength back to its initial value, B_0, again at a rate that is slow compared to the gyro-frequency of the trapped particles but fast compared with the collision frequency. What are $P_\perp$ and $P_\parallel$ now?

(c) Again, wait long enough for the temperatures to equilibrate. Show that the final isotropic pressure is given by

$$P = \left[\frac{2 + 5(B_1/B_0) + 2(B_1/B_0)^2}{9(B_1/B_0)}\right] P_0.$$

(d) For $B_1 = 2B_0$, the above formula gives an $\simeq$11% increase in the thermal energy of the plasma. Repeating this cycle 7 times more than doubles the thermal energy! Where did this thermal energy come from? If you had increased/decreased the field strength on a timescale much longer than the collisional equilibration time, how would P have changed?