1. **A mechanical Alfvén wave.** Suppose we have a perfectly conducting rectangular loop of height h and part of its width x immersed in a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ oriented out of the page. The loop has a mass m and inductance L. Ignore gravity.

(a) Give the loop an initial velocity $\boldsymbol{v} = v_0\hat{\boldsymbol{x}}$ to the right, so that the flux through the loop increases in time. What happens? Describe the motion in words.

According to Lenz's law, the increased magnetic flux threading the loop will generate a clockwise current flowing around the loop. Associated with this current is an electric field $E = vB$, and thus a voltage vBh. Energy will be transferred back and forth between the kinetic energy of the wire, mv^2, and the energy of the inductor, $LI^2/2$. The wire will thus oscillate.

(b) Solve for the motion analytically.

Kirchoff's law gives $v(t)Bh - L\dot{I}(t) = 0$. Energy conservation gives $mv_0^2 = mv^2(t) + LI^2(t)$. Combining these, we have

$$vBh = L\dot{I} = -\frac{v\sqrt{Lm}}{\sqrt{v_0^2 - v^2}}\frac{\mathrm{d}v}{\mathrm{d}t} \quad\Longrightarrow\quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{Bh}{\sqrt{Lm}}\sqrt{v_0^2 - v^2}$$

$$\Longrightarrow\quad \int_{v_0}^{v}\frac{\mathrm{d}v'}{\sqrt{v_0^2 - v'^2}} = -\frac{Bh}{\sqrt{Lm}}\int_0^t \mathrm{d}t' \quad\Longrightarrow\quad v(t) = v_0\sin\left(\frac{Bht}{\sqrt{Lm}}\right).$$

Alternatively, the force on the wire is IhB, and so

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = IhB \quad\Longrightarrow\quad m\frac{\mathrm{d}^2v}{\mathrm{d}t^2} = \frac{\mathrm{d}I}{\mathrm{d}t}hB = \frac{B^2h^2}{L}v \quad\Longrightarrow\quad \frac{\mathrm{d}^2v}{\mathrm{d}t^2} = \frac{B^2h^2}{mL}v,$$

which also gives a harmonic oscillator of frequency $Bh/\sqrt{Lm}$.

(c) Now suppose that the loop has some resistance R. How big should R be before resistance plays an appreciable role in the motion?

Resistance will play a role in the motion once

$$v(t)Bh \sim L\dot{I}(t) \sim IR \quad\Longrightarrow\quad R \sim Bh\sqrt{\frac{L}{m}}.$$

2. **Energy conservation in MHD.** In Prof. Kunz's lecture notes on hydrodynamics, an equation was derived for the evolution of the total energy density (see (II.20)):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2+e+\rho\Phi\right)+\boldsymbol{\nabla\cdot}\left[\left(\frac{1}{2}\rho u^2+\gamma e+\rho\Phi\right)\boldsymbol{u}\right]=\rho\frac{\partial\Phi}{\partial t},\tag{1}$$

where $e=P/(\gamma-1)$, Φ is the gravitational potential, and the other symbols have their usual meanings. Following Prof. Brown's lecture on ideal MHD, which presented the ideal-MHD induction equation,

$$\frac{\partial\boldsymbol{B}}{\partial t}=-c\boldsymbol{\nabla}\times\boldsymbol{E}=\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}),\tag{2}$$

generalize the conservation law (1) to account for the evolution of the magnetic energy density, $B^2/8\pi$. In particular, demonstrate (a) that the magnetic energy is transported by the Poynting flux $\boldsymbol{S}\doteq c\boldsymbol{E}\times\boldsymbol{B}/4\pi$, and (b) that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2+e+\rho\Phi+\frac{B^2}{8\pi}\right)+\boldsymbol{\nabla\cdot}\left[\left(\frac{1}{2}\rho u^2+\gamma e+\rho\Phi\right)\boldsymbol{u}+\boldsymbol{S}\right]=\rho\frac{\partial\Phi}{\partial t}.\tag{3}$$

Dot the induction equation with $\boldsymbol{B}/4\pi$:

$$\begin{aligned}\frac{\partial}{\partial t}\frac{B^2}{8\pi}&=\frac{\boldsymbol{B}}{4\pi}\boldsymbol{\cdot}\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})=\frac{B_i}{4\pi}\epsilon_{ijk}\frac{\partial}{\partial x_j}(\boldsymbol{u}\times\boldsymbol{B})_k\\&=\epsilon_{ijk}\frac{\partial}{\partial x_j}\left[\frac{B_i}{4\pi}(\boldsymbol{u}\times\boldsymbol{B})_k\right]-\epsilon_{ijk}(\boldsymbol{u}\times\boldsymbol{B})_k\frac{\partial}{\partial x_j}\frac{B_i}{4\pi}\\&=\epsilon_{ijk}\frac{\partial}{\partial x_j}\left[\frac{B_i}{4\pi}(\boldsymbol{u}\times\boldsymbol{B})_k\right]-\epsilon_{ijk}\epsilon_{k\ell m}u_\ell B_m\frac{\partial}{\partial x_j}\frac{B_i}{4\pi}\\&=\epsilon_{ijk}\frac{\partial}{\partial x_j}\left[\frac{B_i}{4\pi}(\boldsymbol{u}\times\boldsymbol{B})_k\right]-(\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell})u_\ell B_m\frac{\partial}{\partial x_j}\frac{B_i}{4\pi}\\&=-\boldsymbol{\nabla\cdot}\left[\frac{\boldsymbol{B}\times(\boldsymbol{u}\times\boldsymbol{B})}{4\pi}\right]-\frac{\boldsymbol{uB:\nabla B}}{4\pi}+\boldsymbol{u\cdot\nabla}\frac{B^2}{8\pi}\end{aligned}$$

$$\Longrightarrow\quad\frac{\partial}{\partial t}\frac{B^2}{8\pi}+\boldsymbol{\nabla\cdot S}=-\frac{\boldsymbol{uB:\nabla B}}{4\pi}+\boldsymbol{u\cdot\nabla}\frac{B^2}{8\pi}.$$

Now we need the additional magnetic terms in the kinetic energy equation. To obtain those, dot the Lorentz force with $\boldsymbol{u}$:

$$\boldsymbol{u\cdot}\left(-\boldsymbol{\nabla}\frac{B^2}{8\pi}+\frac{\boldsymbol{B\cdot\nabla B}}{4\pi}\right).$$

But this is just minus the right-hand side of our magnetic energy equation. So, adding the total hydrodynamic energy equation including these Lorentz-force terms to the magnetic energy equation leads to

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2+e+\rho\Phi+\frac{B^2}{8\pi}\right)+\boldsymbol{\nabla\cdot}\left[\left(\frac{1}{2}\rho u^2+\gamma e+\rho\Phi\right)\boldsymbol{u}+\boldsymbol{S}\right]=\rho\frac{\partial\Phi}{\partial t}.$$

3. **Transport of energy by a circularly polarized Alfvén wave.** A circularly polarized Alfvén wave of amplitude $\delta B_\perp$ propagates along an otherwise uniform magnetic field $B_0\hat{\boldsymbol{z}}$:

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta B_\perp\,\boldsymbol{e}_\perp(t,z) \quad \text{and} \quad \boldsymbol{u} = -\frac{\delta B_\perp}{\sqrt{4\pi\rho}}\,\boldsymbol{e}_\perp(t,z), \tag{4}$$

where

$$\boldsymbol{e}_\perp(t,z) = \cos[k(v_{\rm A}t - z)]\hat{\boldsymbol{x}} + \sin[k(v_{\rm A}t - z)]\hat{\boldsymbol{y}}. \tag{5}$$

(a) Draw the magnetic-field line at $t = 0$. Which way is the wave propagating? Is the wave right-handed or left-handed?

The wave is propagating in the z direction, and it is right-handed.

(b) Prove that the magnetic-field strength B is a constant, despite the presence of the wave.

$$B^2 = B_0^2 + 2B_0\,\delta B_\perp(\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{e}_\perp) + \delta B_\perp^2\,(\boldsymbol{e}_\perp\boldsymbol{\cdot}\boldsymbol{e}_\perp) = B_0^2 + \delta B_\perp^2.$$

(c) Show that (4) is an exact nonlinear solution of the ideal-MHD equations.

We already know that Alfvén waves are linear solutions to the ideal-MHD equations. So we need only examine the nonlinear terms, all of which either involve $\boldsymbol{\nabla}B^2 = 0$ or $\boldsymbol{e}_\perp\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{e}_\perp = 0$.

(d) Calculate the time-averaged Poynting flux $\langle\boldsymbol{S}\rangle_t \doteq \langle c\boldsymbol{E}\times\boldsymbol{B}/4\pi\rangle_t$ for this wave. Write it in terms of the total wave energy $\mathcal{E} = \rho u^2/2 + \delta B_\perp^2/8\pi$. Interpret your result physically.

$$\begin{aligned}
\langle\boldsymbol{S}\rangle_t = \left\langle \frac{\boldsymbol{B}\times(\boldsymbol{u}\times\boldsymbol{B})}{4\pi}\right\rangle_t &= \left\langle \boldsymbol{u}\frac{B^2}{4\pi} - \boldsymbol{u}\boldsymbol{\cdot}\frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right\rangle_t \\
&= \left\langle -\boldsymbol{e}_\perp\frac{\delta B_\perp}{\sqrt{4\pi\rho}}\frac{B_0^2+\delta B_\perp^2}{4\pi} + \frac{\delta B_\perp^2}{\sqrt{4\pi\rho}}\frac{B_0\hat{\boldsymbol{z}}+\delta B_\perp\boldsymbol{e}_\perp}{4\pi}\right\rangle_t \\
&= \left\langle -\boldsymbol{e}_\perp\frac{\delta B_\perp}{\sqrt{4\pi\rho}}\frac{B_0^2}{4\pi} + \hat{\boldsymbol{z}}\frac{\delta B_\perp^2}{\sqrt{4\pi\rho}}\frac{B_0}{4\pi}\right\rangle_t \\
&= v_{\rm A0}\,\frac{\delta B_\perp^2}{4\pi}\,\hat{\boldsymbol{z}} \doteq v_{\rm A0}\,\mathcal{E}\,\hat{\boldsymbol{z}}.
\end{aligned}$$

Energy is transported along the mean magnetic field at the Alfvén speed.

4. **Energy conservation in MHD turbulence.** In this problem, you will derive a conservation law for an *incompressible* (i.e., $\nabla \cdot \boldsymbol{u} = 0$) turbulent fluid.

(a) The MHD induction equation including a constant magnetic resistivity η is

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + \frac{c^2\eta}{4\pi}\nabla^2\boldsymbol{B}, \tag{6}$$

where $\mathrm{D}/\mathrm{D}t \doteq \partial/\partial t + \boldsymbol{u} \cdot \nabla$ is the Lagrangian derivative. Dot this equation with $\boldsymbol{B}$ and integrate over real space to derive an evolution equation for the magnetic energy $\int \mathrm{d}^3\boldsymbol{r}\, |\boldsymbol{B}|^2/8\pi$. Assume zero flux at the boundaries (taken to be at infinity). (You may use the answer from Problem #2 and then add on the appropriate resistive contribution.)

From the solution to Problem #2, in ideal MHD the magnetic energy satisfies

$$\frac{\partial}{\partial t}\frac{B^2}{8\pi} + \nabla \cdot \boldsymbol{S} = -\frac{\boldsymbol{uB} : \nabla\boldsymbol{B}}{4\pi} + \boldsymbol{u} \cdot \nabla\frac{B^2}{8\pi} = -\frac{\boldsymbol{uB} : \nabla\boldsymbol{B}}{4\pi} + \nabla \cdot \left(\frac{B^2}{8\pi}\boldsymbol{u}\right),$$

where we have used $\nabla \cdot \boldsymbol{u} = 0$ to obtain the last equality. Integrating this equation over all space and assuming zero flux at the boundaries yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}\, \frac{B^2}{8\pi} = -\int \mathrm{d}^3\boldsymbol{r}\, \frac{\boldsymbol{uB} : \nabla\boldsymbol{B}}{4\pi}.$$

We add to this the resistive losses:

$$\int \mathrm{d}^3\boldsymbol{r}\, \frac{\boldsymbol{B}}{4\pi} \cdot \frac{c^2\eta}{4\pi}\nabla^2\boldsymbol{B} = -\int \mathrm{d}^3\boldsymbol{r}\, \frac{c^2\eta}{4\pi}\left|\frac{\nabla\boldsymbol{B}}{4\pi}\right|^2 = -\int \mathrm{d}^3\boldsymbol{r}\, \eta|\boldsymbol{j}|^2,$$

i.e., Joule heating ("I^2R" in the language of high-school electronics).

(b) The MHD momentum equation including a constant dynamical viscosity μ is

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}}{4\pi} + \mu\nabla^2\boldsymbol{u}. \tag{7}$$

Dot this equation with $\boldsymbol{u}$ and integrate over real space to derive an evolution equation for the kinetic energy $\int \mathrm{d}^3\boldsymbol{r}\, \rho|\boldsymbol{u}|^2/2$. Again, assume zero flux at the boundaries (taken to be at infinity). At some point you'll need the mass continuity equation for an incompressible fluid,

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{u} \cdot \nabla\rho.$$

(Note: this is done in Prof. Kunz's hydrodynamics lecture notes for an inviscid fluid, so you need only figure out the appropriate viscous contribution.)

Minding the note, we need only compute the viscous losses:

$$\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{u} \cdot \mu\nabla^2\boldsymbol{u} = -\int \mathrm{d}^3\boldsymbol{r}\, \mu|\nabla\boldsymbol{u}|^2 = -\int \mathrm{d}^3\boldsymbol{r}\, \mu|\boldsymbol{\omega}|^2,$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is the flow vorticity. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}\, \frac{1}{2}\rho u^2 = \int \mathrm{d}^3\boldsymbol{r}\, \frac{\boldsymbol{uB} : \nabla\boldsymbol{B}}{4\pi} - \int \mathrm{d}^3\boldsymbol{r}\, \mu|\boldsymbol{\omega}|^2.$$

(c) Add the results of parts (a) and (b) together to obtain the following conservation law for the total mechanical (magnetic + kinetic) energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}\left(\frac{|\boldsymbol{B}|^2}{8\pi}+\frac{\rho|\boldsymbol{u}|^2}{2}\right) = -\int \mathrm{d}^3\boldsymbol{r}\left(\eta|\boldsymbol{j}|^2+\mu|\boldsymbol{\omega}|^2\right) \leq 0, \tag{8}$$

where $\boldsymbol{j} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$ is the current density and $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ is the flow vorticity. Suppose there were a source term injecting mechanical energy into the system at large scales. Explain physically how a steady state ($\mathrm{d}/\mathrm{d}t = 0$) might be achieved. (Hint: what if η and μ were really, really small – would it matter?)

Combining the results from parts (a) and (b), the $\boldsymbol{uB}:\boldsymbol{\nabla B}$ terms cancel and we obtain (8). To obtain steady state with a source term, the energy would have to be dissipated. If η and μ were really small, this would require small-scale structure to be generated in the magnetic field and/or velocity field. This is just what a turbulent cascade does.

5. **Kelvin's circulation theorem** is an extremely important result in fluid dynamics. Every time you ride an airplane, you owe your life to it. In this problem you will prove it, as well as investigate the effects of baroclinicity and the Lorentz force.

(a) Start by taking the curl of the MHD force equation

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho},$$

where $\boldsymbol{j} = (c/4\pi)\boldsymbol{\nabla}\times\boldsymbol{B}$ is the current density, to obtain an evolution equation for the *vorticity* $\boldsymbol{\omega} \doteq \boldsymbol{\nabla}\times\boldsymbol{u}$. Use a particular vector identity to write it in the form $\partial\boldsymbol{\omega}/\partial t = \boldsymbol{\nabla}\times(\dots)$. This should look *almost* like the ideal-MHD induction equation, but not quite. (The resolution of this "not quite" involves freezing the magnetic field in the electron fluid and retaining a non-ideal term in the induction equation that allows the magnetic field to drift through the ion species – the so-called "Hall effect".)

Note: The vorticity is divergence free, which means that vortex lines cannot end within the fluid – they must either close on themselves (like a smoke ring) or intersect a boundary (like a tornado). Any fresh vortex lines that are made must be created as continuous curves that grow out of points or lines were the vorticity vanishes.

The curl of the ideal-MHD momentum equation is

$$\boldsymbol{\nabla}\times\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho} - \boldsymbol{\nabla}\Phi\right)$$

$$\Longrightarrow \quad \frac{\partial\boldsymbol{\omega}}{\partial t} + \underbrace{\boldsymbol{\nabla}\times\big[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\big]}_{=-\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{\omega})} = \boldsymbol{\nabla}\times\left(-\frac{1}{\rho}\boldsymbol{\nabla}P + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho}\right)$$

$$\boxed{\frac{\partial\boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla}\times\left(\boldsymbol{u}\times\boldsymbol{\omega} - \frac{1}{\rho}\boldsymbol{\nabla}P + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho}\right)}$$

(b) The *circulation* is defined by

$$\Gamma \doteq \oint_{\partial\mathcal{S}} \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{\ell}, \tag{9}$$

where $\mathrm{d}\boldsymbol{\ell}$ is an infinitesimal line element along a simple closed contour $\partial\mathcal{S}(t)$ bounding a material surface $\mathcal{S}(t)$ moving with velocity $\boldsymbol{u}$. By Stokes' theorem, this is equivalent to

$$\Gamma = \int_{\mathcal{S}} \boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{\mathcal{S}},$$

which states that the circulation around the boundary $\partial\mathcal{S}$ can be calculated as the number of vortex lines that thread the enclosed area $\mathcal{S}$. Take $\mathrm{D}/\mathrm{D}t$ of (9) and use the result of part (a) to obtain an equation for the evolution of the circulation. (Hint: don't forget to compute the time rate of change of the area, something that was done in the MHD lecture when proving Alfvén's theorem.)

Note: "simple closed contour" means *simply connected* – that is, the region must be such that we can shrink the contour to a point without leaving the region. A region with a hole (like a bathtub drain) is *not* simply connected.

The circulation Γ evolves according to

$$\begin{aligned}
\frac{\mathrm{D}\Gamma}{\mathrm{D}t} &= \frac{\mathrm{D}}{\mathrm{D}t}\oint_{\partial S} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{\ell} \\
&= \oint_{\partial S} \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\cdot \mathrm{d}\boldsymbol{\ell} + \oint_{\partial S} \boldsymbol{u}\cdot\frac{\mathrm{D}\mathrm{d}\boldsymbol{\ell}}{\mathrm{D}t} \\
&= \oint_{\partial S} \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\cdot \mathrm{d}\boldsymbol{\ell} + \oint_{\partial S} \boldsymbol{u}\cdot\left[(\mathrm{d}\boldsymbol{\ell}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] \\
&= \oint_{\partial S}\Big(\underbrace{-\frac{1}{\rho}\boldsymbol{\nabla}P}_{\substack{=0 \text{ if} \\ p=p(\rho)}} + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho} - \underbrace{\boldsymbol{\nabla}\Phi}_{=0}\Big)\cdot\mathrm{d}\boldsymbol{\ell} + \cancelto{0}{\oint_{\partial S}\mathrm{d}\boldsymbol{\ell}\cdot\boldsymbol{\nabla}\frac{u^2}{2}}
\end{aligned}$$

$$\Longrightarrow \quad \boxed{\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \oint_{\partial S}\left(-\frac{\mathrm{d}P}{\rho}\right) + \oint_{\partial S}\left(\frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho}\right)\cdot\mathrm{d}\boldsymbol{\ell}}$$

Alternatively,

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\int_S \boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{S} &= \int_S \frac{\partial\boldsymbol{\omega}}{\partial t}\cdot\mathrm{d}\boldsymbol{S} + \oint_{\partial S}\boldsymbol{\omega}\cdot(\boldsymbol{u}\times\mathrm{d}\boldsymbol{\ell}) \\
&= \int_S \frac{\partial\boldsymbol{\omega}}{\partial t}\cdot\mathrm{d}\boldsymbol{S} + \oint_{\partial S}(\boldsymbol{\omega}\times\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{\ell} \\
&= \int_S \frac{\partial\boldsymbol{\omega}}{\partial t}\cdot\mathrm{d}\boldsymbol{S} + \int_S \boldsymbol{\nabla}\times(\boldsymbol{\omega}\times\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{S} \\
&= \int_S\left[\cancel{\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{\omega})} + \boldsymbol{\nabla}\times\left(-\frac{1}{\rho}\boldsymbol{\nabla}P + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho}\right) + \cancel{\boldsymbol{\nabla}\times(\boldsymbol{\omega}\times\boldsymbol{u})}\right]\cdot\mathrm{d}\boldsymbol{S},
\end{aligned}$$

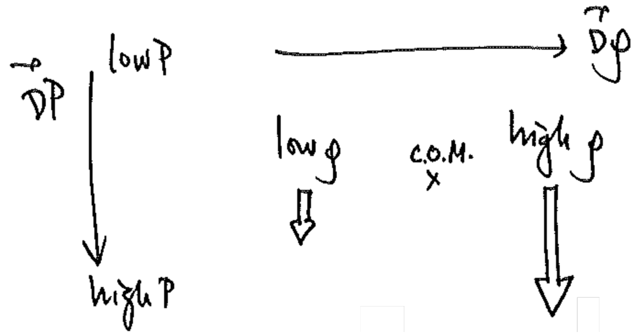
which gives the same thing.

(c) Answer all of the following questions. Under what conditions is the circulation conserved? Why would pressure gradients and density gradients have anything to do with generating circulation? Draw a picture as part of your explanation. (Hint: It might help to imagine what would happen in an atmosphere if the pressure gradient were being imposed by vertical gravity and the density of air were greater in the east than the west.) Why would the Lorentz force have anything to do with generating circulation? (Hint: Take an irrotational fluid and thread it with a twisted magnetic field. Let it go. What would happen?) Would it help or hurt vorticity conservation if the magnetic field weren't perfectly frozen into the plasma? Why?

The circulation is conserved if and only if $P = P(\rho)$ *and*

$$\oint_{\partial S}\left(\frac{\boldsymbol{j}\times\boldsymbol{B}}{c\rho}\right)\cdot\mathrm{d}\boldsymbol{\ell} = 0.$$

The former constraint, which may be written $\boldsymbol{\nabla}P\times\boldsymbol{\nabla}\rho = 0$ – "zero baroclinicity" or "barotropic" – may be understood visually:



The force is greater on the fluid element with larger density (think gravity $\rho \boldsymbol{g} = -\boldsymbol{\nabla} P$ pulling down), and so there is a torque about the center of mass (c.o.m.). This causes rotation, thus vorticity.

The Lorentz force gives rotation because magnetic-field lines resist twisting. Imagine threading a fluid with twisted field lines (i.e., a current). Those lines would unwind and, by flux freezing, would carry the fluid with them, spinning it. If the magnetic field weren't frozen into the plasma, then the field could unwind without pulling the fluid with it, and no vorticity would be generated.

(d) Knowing that the Coriolis force is $-2\boldsymbol{\Omega}\times\boldsymbol{u}$, prove that the circulation in a rotating reference frame is given by $\Gamma + \int_S 2\boldsymbol{\Omega}\cdot \mathrm{d}\boldsymbol{S}$, where $\boldsymbol{\Omega}$ is the angular velocity.

Define the velocity as measured in a rotating frame, $\boldsymbol{u}_{\mathrm{rot}} = \boldsymbol{u} + \boldsymbol{\Omega}\times\boldsymbol{r}$. Then the associated vorticity in the rotating frame is

$$\boldsymbol{\omega}_{\mathrm{rot}} = \boldsymbol{\omega} + \boldsymbol{\nabla}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) = \boldsymbol{\omega} + \boldsymbol{\Omega}(\boldsymbol{\nabla}\cdot\boldsymbol{r}) - (\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{r} = \boldsymbol{\omega} + 3\boldsymbol{\Omega} - \boldsymbol{\Omega} = \boldsymbol{\omega} + 2\boldsymbol{\Omega}.$$

The associated circulation is

$$\begin{aligned}\Gamma_{\mathrm{rot}} &= \int_S \boldsymbol{\omega}_{\mathrm{rot}}\cdot \mathrm{d}\boldsymbol{S} = \int_S (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\cdot \mathrm{d}\boldsymbol{S} \\ &= \oint_{\partial S} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{\ell} + \int_S 2\boldsymbol{\Omega}\cdot \mathrm{d}\boldsymbol{S} \\ &= \Gamma + \int_S 2\boldsymbol{\Omega}\cdot \mathrm{d}\boldsymbol{S}.\end{aligned}$$

Alternatively, you could look at the Coriolis force in the momentum equation and compute its line integral about the surface of a fluid element:

$$\begin{aligned}\oint_{\partial S} \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\cdot \mathrm{d}\boldsymbol{\ell} &= \cdots - \oint_{\partial S} (2\boldsymbol{\Omega}\times\boldsymbol{u})\cdot \mathrm{d}\boldsymbol{\ell} \\ &= \cdots - \oint_{\partial S} 2\boldsymbol{\Omega}\cdot(\boldsymbol{u}\times \mathrm{d}\boldsymbol{\ell}) \\ &= \cdots - \frac{\mathrm{D}}{\mathrm{D}t}\int_S 2\boldsymbol{\Omega}\cdot \mathrm{d}\boldsymbol{S}.\end{aligned}$$

6. **Magnetorotational Instability with springs**. The acknowledgement at the end of Balbus & Hawley (1992a) reads, "It is fitting and proper to acknowledge Alar Toomre for this important insight that the Hill equations had something to contribute to the MHD stability problem." This insight is what led Balbus and Hawley to develop the now-famous spring model of the MRI, which was then used to conjecture that the Oort A-value is the universal growth rate limit for accretion-disk shear instabilities. The Hill equations describe local disk dynamics in a rotating frame – *local* in that they describe small excursions $x \doteq R - R_0$ and $y \doteq R_0(\varphi - \Omega_0 t)$ from a circular orbit $R = R_0$, $\varphi = \Omega_0 t$. They are given by:

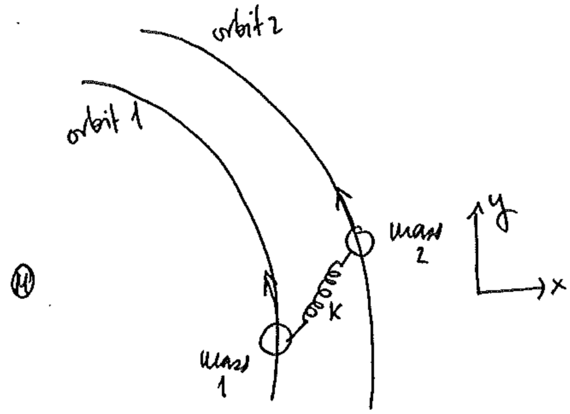
$$\ddot{x} - 2\Omega_0 \dot{y} = -4A_0\Omega_0 x + f_x, \tag{10a}$$
$$\ddot{y} + 2\Omega_0 \dot{x} = f_y, \tag{10b}$$

where the overdot indicates a time derivative and f_x and f_y represent local forces in the x and y directions. The Oort A-value $A_0 = -(3/4)\Omega_0$ for Keplerian rotation.[1]

The MRI analogy goes as follows. Consider the local force to be nondissipative and to act by restoring a displacement back to its equilibrium position. The leading-order contribution to f_x and f_y in a Taylor expansion about $(R_0, \Omega_0 t)$ is linear; for an *isotropic* force, we have $f_x = -Kx$ and $f_y = -Ky$, where $K > 0$ is some constant. (You could also profitably think of this force as being due to an ideal spring with spring constant K.) Then (10) becomes

$$\ddot{x} - 2\Omega_0 \dot{y} = -4A_0\Omega_0 x - Kx, \tag{11a}$$
$$\ddot{y} + 2\Omega_0 \dot{x} = -Ky. \tag{11b}$$

Visually,



Now then...

(a) For small displacements x, y, show that the solutions are $\propto \exp(\pm \mathrm{i}\omega t)$ with

$$\omega^4 - \omega^2\big(\kappa^2 + 2K\big) + K\big(K + 4A_0\Omega_0\big) = 0, \tag{12}$$

where $\kappa^2 \doteq 4\Omega_0^2(1 + A_0/\Omega_0)$ is the square of the epicyclic frequency, which is positive for Keplerian rotation. Equation (12) should look familiar from the lecture notes on

[1]The notation for differential rotation varies in the accretion-disk literature; here's a dictionary: $2A_0 = -q\Omega_0 = \mathrm{d}\Omega/\mathrm{d}\ln R|_{R=R_0}$. Often, the "0" subscript is simply dropped for ease of notation.

MHD instabilities: set $K = 0$ and you get trivial displacements ($\omega^2 = 0$) and epicycles ($\omega^2 = \kappa^2$); replace K with $(\boldsymbol{k} \cdot \boldsymbol{v}_{\rm A})^2$ and you get the axisymmetric MRI linear dispersion relation. Show that $A_0 < 0$ is a necessary (but not sufficient) condition for instability.

Equations (11) admit solutions $\propto \exp(-{\rm i}\omega t)$, resulting in the following system:

$$\begin{bmatrix} -\omega^2 + K + 4A_0\Omega_0 & 2\Omega_0 {\rm i}\omega \\ -2\Omega_0 {\rm i}\omega & -\omega^2 + K \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Taking the determinant of the 2×2 matrix and setting it to zero yields the dispersion relation

$$\omega^4 - \omega^2\big(2K + \kappa^2\big) + K\big(K + 4A_0\Omega_0\big) = 0,$$

where $\kappa^2 \doteq 4\Omega_0^2(1 + A_0/\Omega_0)$ is the square of the epicyclic frequency. Write $\gamma = -{\rm i}\omega$, which is a growth rate. The solution to this bi-quadratic that might grow is

$$\gamma^2 = -\left(\frac{2K + \kappa^2}{2}\right) + \left[\left(\frac{2K + \kappa^2}{2}\right)^2 - K\big(K + 4A_0\Omega_0\big)\right]^{1/2}.$$

In order for the term in the square root to overwhelm the first (negative) term and lead to growth, we require

$$K + 4A_0\Omega_0 < 0.$$

A necessary condition for instability is thus $A_0 < 0$, an outwardly decreasing angular velocity.

(b) S. A. Balbus and J. F. Hawley, *Astrophys. J.* **392**, 662 (1992) conjecture "that the Oort A-value is an upper bound to the growth rate of any instability feeding upon the free energy of differential rotation." En route, they show that the maximum growth rate of the MRI is the Oort-A value, that it occurs at $K_{\rm max}/\Omega_0^2 = -(A_0/\Omega_0)(2 + A_0/\Omega_0)$, and that the corresponding eigenvector satisfies $y/x = -1$, i.e., radial and azimuthal displacements are equal in size. Prove these three facts.

Differentiate the dispersion relation with respect to K and set $\partial\omega^2/\partial K = 0$:

$$-2\omega^2 + 2K + 4A_0\Omega = 0 \quad \Longrightarrow \quad K = \omega^2 - 2A_0\Omega_0.$$

Substitute this back into the dispersion relation to obtain the maximum growth rate $\gamma^2_{\rm max} = -\omega^2_{\rm max} = A_0^2$. Inserting this value into $K = \omega^2 - 2A_0\Omega_0$ gives the desired $K_{\rm max}/\Omega_0^2 = -(A_0/\Omega_0)(2 + A_0/\Omega_0)$. Returning to the system of equations, we see that $y/x = 2\Omega_0{\rm i}\omega/(K - \omega^2)$. Inserting these maximum values gives $(y/x)_{\rm max} = -1$.

(c) Use these to show that, at maximum growth, the Lagrangian change in the rotation frequency of a displaced fluid element is $\Delta\Omega = \dot{y}/R_0 = -|A_0|x/R_0$ and that the corresponding Lagrangian change in its specific angular momentum $\ell = \Omega R^2$ satisfies

$$\frac{\Delta\ell}{\ell_0} = 2\frac{x}{R_0} + \frac{\Delta\Omega}{\Omega_0} = 2\left(1 - \frac{|A_0|}{2\Omega_0}\right)\frac{x}{R_0}. \tag{13}$$

Then show that outwardly (inwardly) displaced fluid elements always have more (less) angular momentum that the orbits they are passing through (which is what makes instability possible). (Hint: what is the difference in ℓ between two undisturbed orbits a radial distance x apart, in a disk in which $\mathrm{d}\Omega/\mathrm{d}\ln R = 2A_0 < 0$?)

$$\frac{\Delta\Omega}{\Omega_0} = \frac{\dot{y}}{\Omega_0 R_0} \to -\frac{\mathrm{i}\omega_{\max}}{\Omega_0}\left(\frac{y}{x}\right)_{\max}\frac{x}{R_0} = -\frac{|A_0|}{\Omega_0}\frac{x}{R_0}$$

$$\frac{\Delta\ell}{\ell_0} = \frac{\Delta\Omega}{\Omega_0} + 2\frac{x}{R_0} \to \left(2 - \frac{|A_0|}{\Omega_0}\right)\frac{x}{R_0}.$$

Note that the difference between the angular momentum of an orbit at $R_0 + x$ and one at R_0 is

$$\frac{\Delta\ell}{\ell_0} = \left(1 + \frac{x}{R_0}\right)^{2(1+A_0/\Omega_0)} - 1 \approx \left(2 - 2\frac{|A_0|}{\Omega_0}\right)\frac{x}{R_0},$$

assuming $A_0 < 0$. Thus, outwardly displaced fluid elements always have more angular momentum than the orbit they are passing through.

(d) **Bonus**. Set $f_x = -K_x x$ and $f_y = -K_y y$ with $K_x \neq K_y$ being positive constants. Compute the new dispersion relation governing the time-evolution of small displacements. Is the growth rate larger or smaller than the Oort-A value for $K_x > K_y$? for $K_x < K_y$? From this result, find the maximum growth rate $\gamma_{\max}$ and the (hint: asymptotic) values of K_x and K_y at which $\gamma_{\max}$ is achieved. (It may help to make a quick contour plot of the growth rate in the K_x–K_y plane using your dispersion relation.) E. Quataert, W. Dorland, and G. W. Hammett *Astrophys. J.* **577**, 524 (2002) used this as a model for the magnetorotational instability in a collisionless plasma.

The Hill equations with an anisotropic spring are

$$\ddot{x} - 2\Omega_0\dot{y} = -4A_0\Omega_0 x - K_x x,$$
$$\ddot{y} + 2\Omega_0\dot{x} = -K_y y,$$

with $K_x \neq K_y$ being positive constants. Adopting solutions $x, y \sim \exp(-\mathrm{i}\omega t)$, we have

$$\begin{bmatrix} -\omega^2 + K_x + 4A_0\Omega_0 & 2\Omega_0\mathrm{i}\omega \\ -2\Omega_0\mathrm{i}\omega & -\omega^2 + K_y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Taking the determinant of the 2×2 matrix and setting it to zero yields the dispersion relation

$$\omega^4 - \omega^2\big(K_x + K_y + \kappa^2\big) + K_y\big(K_x + 4A_0\Omega_0\big) = 0,$$

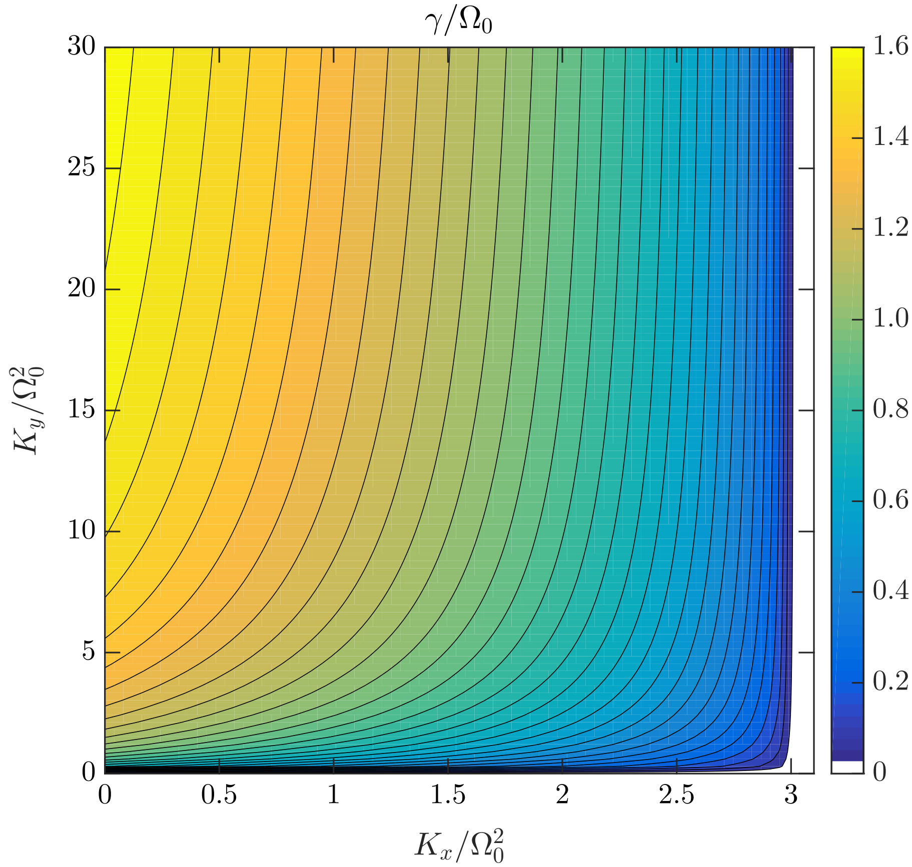
where $\kappa^2 \doteq 4\Omega_0^2(1 + A_0/\Omega_0)$ is the square of the epicyclic frequency. Write $\gamma = -\mathrm{i}\omega$, which is a growth rate. Solving the bi-quadratic leads to

$$\gamma^2 = -\left(\frac{K_x + K_y + \kappa^2}{2}\right) + \left[\left(\frac{K_x + K_y + \kappa^2}{2}\right)^2 - K_y\big(K_x + 4A_0\Omega_0\big)\right]^{1/2}.$$

Because the final term in brackets must be negative for the possibility of growth, having $K_y > K_x$ increases the growth rate (assuming $A_0 < 0$). In fact, the maximum growth rate occurs at $K_y \gg \Omega_0^2 \gg K_x$:

$$\gamma_{\max}^2 \approx -\frac{K_y}{2} + \frac{K_y}{2}\left(1 - \frac{16A_0\Omega_0}{K_y}\right)^{1/2} \approx -4A_0\Omega_0 \quad\Longrightarrow\quad \gamma_{\max} = \sqrt{|4A_0\Omega_0|}.$$

This is *larger* than the maximum MRI growth rate in MHD! Below is a plot of growth rate in the K_x–K_y plane for a Keplerian disk; note that the maximum is achieved as $K_x \to 0$, $K_y \to \infty$:



It's not part of the solution, but note that the eigenvector here has

$$\frac{y}{x} = \frac{2\Omega_0 \mathrm{i}\omega}{-\omega^2 + K_y} = -\frac{2\Omega_0\gamma}{\gamma^2 + K_y}. \tag{14}$$

With $\gamma \approx \gamma_{\max}$, this is $|y/x| \sim \Omega_0^2/K_y \ll 1$, i.e., a predominantly radial displacement. Following §2.4 of Balbus & Hawley (1992a), the maximum local radial separation rate of nearby undisturbed orbital elements undergoing differential rotation is the Oort-A value. But this is measured along the longitude $l = \pi/4$ (i.e., $y/x = -1$) for any shear flow, *not* $|y/x| \ll 1$. The issue is that the radial and azimuthal couplings are different, and it is ultimately the azimuthal restoring force that removes angular momentum from an inwardly displaced fluid elements and transfers it to a tethered outwardly displaced fluid element. The radial force is stabilizing, since it is trying to pin the element to its unperturbed location.

7. **Drifts in Dipoles**. The equation for a dipole magnetic field in spherical coordinates is given by

$$\boldsymbol{B} = \frac{3\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{r})}{r^5} - \frac{\boldsymbol{m}}{r^3} = \frac{m}{r^3}\left(2\cos\vartheta\,\hat{\boldsymbol{r}} + \sin\vartheta\,\hat{\boldsymbol{\vartheta}}\right), \tag{15}$$

where $\boldsymbol{m} = m\hat{\boldsymbol{z}}$ is the magnetic moment.

(a) Show that the equation for a magnetic-field line is $r = R\sin^2\vartheta$, where R is the radius of the magnetic-field line at the equator ($\vartheta = \pi/2$).

Magnetic-field-line equations are

$$\frac{\mathrm{d}r}{B_r} = \frac{r\mathrm{d}\vartheta}{B_\vartheta} = \frac{r\sin\vartheta\mathrm{d}\varphi}{B_\varphi} = \frac{\mathrm{d}s}{B},$$

where s is the distance along the field line. Thus, for the dipole field,

$$\frac{\mathrm{d}r}{2\cos\vartheta} = \frac{r\mathrm{d}\vartheta}{\sin\vartheta} \quad\Longrightarrow\quad \int_R^{r(\vartheta)}\frac{\mathrm{d}r}{r} = 2\int_{\pi/2}^{\vartheta}\frac{\mathrm{d}(\sin\vartheta)}{\sin\vartheta} \quad\Longrightarrow\quad r(\theta) = R\sin^2\vartheta.$$

(b) Show that the curvature of the magnetic-field line at the equator ($\vartheta = \pi/2$) is $R_\mathrm{c} = R/3$.

$$\hat{\boldsymbol{b}} \doteq \frac{\boldsymbol{B}}{B} = \frac{2\cos\vartheta\,\hat{\boldsymbol{r}} + \sin\vartheta\,\hat{\boldsymbol{\vartheta}}}{\sqrt{3\cos^2\vartheta + 1}} \quad\Longrightarrow\quad \hat{\boldsymbol{b}}(\pi/2) = \hat{\boldsymbol{\vartheta}},\ \left.\frac{\partial b_r}{\partial\vartheta}\right|_{\pi/2} = -2,\ \left.\frac{\partial b_\vartheta}{\partial\vartheta}\right|_{\pi/2} = 0$$

$$\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\hat{\boldsymbol{b}} = \frac{\hat{\boldsymbol{r}}}{r}\frac{\partial b_r}{\partial\vartheta} + \frac{\hat{\boldsymbol{\vartheta}}}{r}\frac{\partial b_\varphi}{\partial\vartheta} - \frac{b_\varphi^2}{r}\hat{\boldsymbol{r}} + \frac{b_r b_\vartheta}{r}\hat{\boldsymbol{\vartheta}} \quad\Longrightarrow\quad \hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\hat{\boldsymbol{b}} = -\frac{3\hat{\boldsymbol{r}}}{R}\ \text{at}\ \vartheta = \frac{\pi}{2}.$$

(c) Compute the curvature drift of a particle with charge q and parallel kinetic energy $W_\parallel$ at a radial distance R at the equator.

The curvature drift is

$$\boldsymbol{v}_\mathrm{c} = -\frac{v_\parallel^2}{\Omega}(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\hat{\boldsymbol{b}})\times\hat{\boldsymbol{b}} = 6W_\parallel\frac{cR^2}{qm}\hat{\boldsymbol{\varphi}}\ \text{at}\ \vartheta = \frac{\pi}{2}.$$

(d) Compute the grad-B drift of a particle with charge q and perpendicular kinetic energy $W_\perp$ at a radial distance R at the equator. For what ratio $W_\perp/W_\parallel$ are the drifts the same?

The grad-B drift is

$$\boldsymbol{v}_{\nabla B} = \frac{v_\perp^2}{2\Omega}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\ln B = 3W_\perp\frac{cR^2}{qm}\hat{\boldsymbol{\varphi}}\ \text{at}\ \vartheta = \frac{\pi}{2}.$$

Drifts are the same for $W_\perp/W_\parallel = 2$.

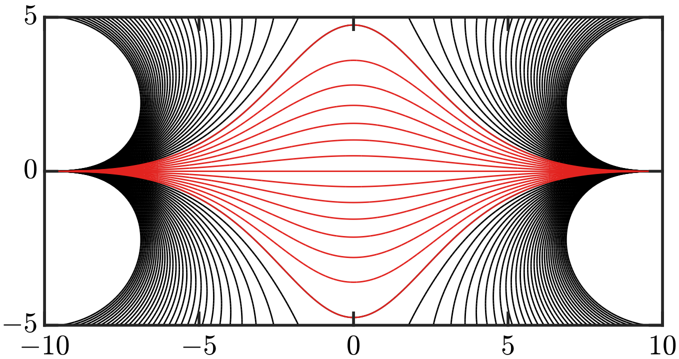
Now suppose there are two aligned magnetic dipoles with moment $\boldsymbol{m}$ spatially separated by $2\boldsymbol{a}$ about the origin. The magnetic field is then given by

$$\boldsymbol{B}(\boldsymbol{r}) = \left[\frac{3\boldsymbol{r}_+(\boldsymbol{m}\cdot\boldsymbol{r}_+)}{r_+^5} - \frac{\boldsymbol{m}}{r_+^3}\right] + \left[\frac{3\boldsymbol{r}_-(\boldsymbol{m}\cdot\boldsymbol{r}_-)}{r_-^5} - \frac{\boldsymbol{m}}{r_-^3}\right], \tag{16}$$

where $\boldsymbol{r}_\pm \doteq \boldsymbol{r} \pm \boldsymbol{a}$. This field may be obtained by taking the curl of the vector potential

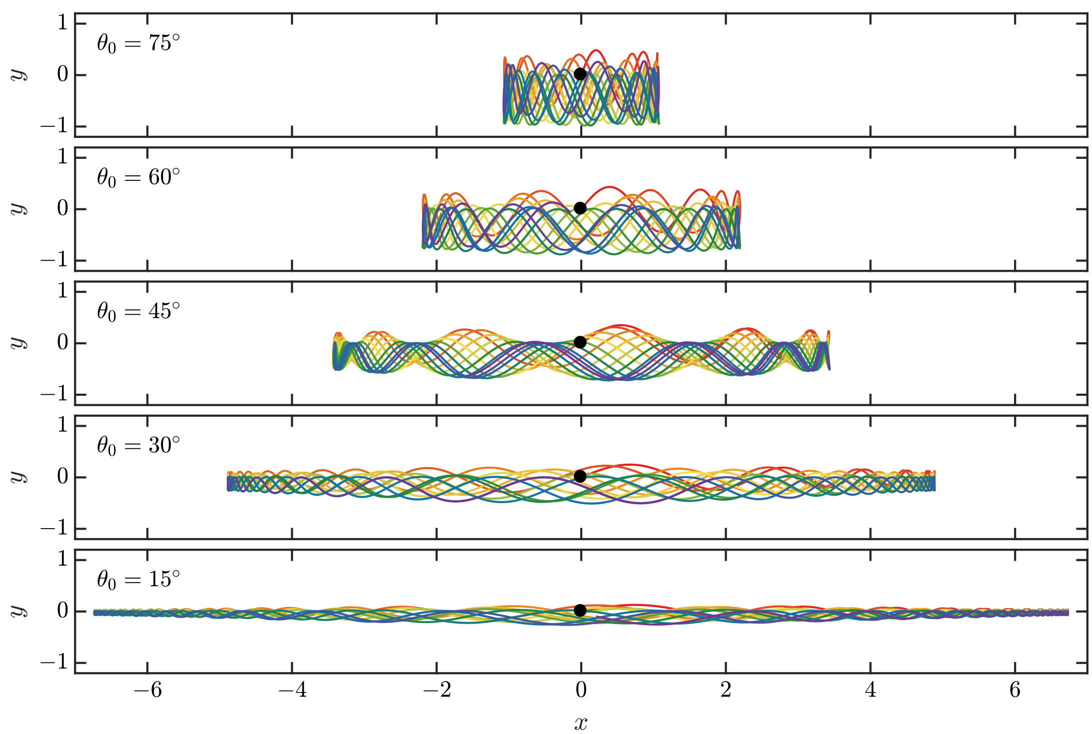
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\boldsymbol{m}\times\boldsymbol{r}_+}{r_+^3} + \frac{\boldsymbol{m}\times\boldsymbol{r}_-}{r_-^3}. \tag{17}$$

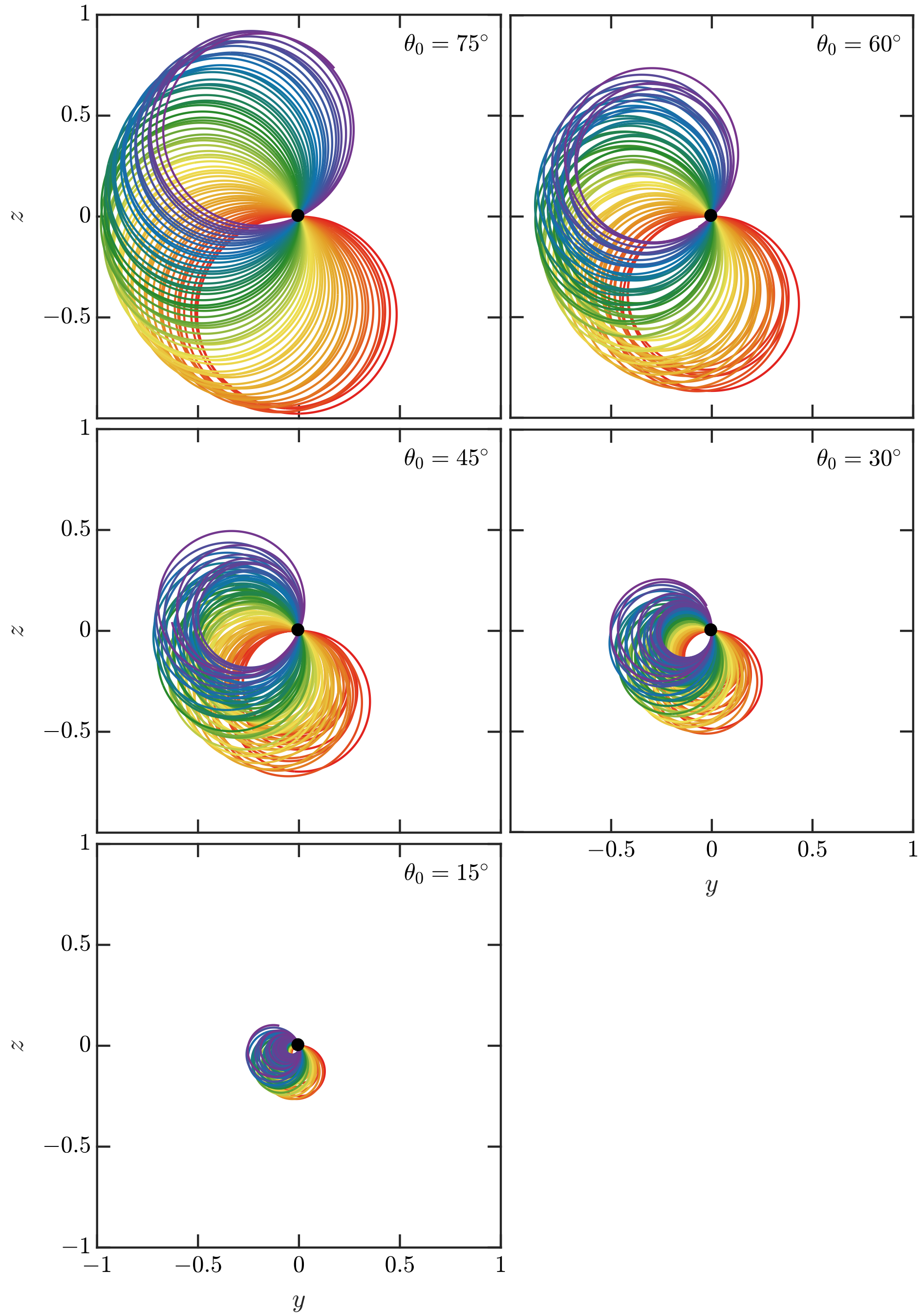
Because $\partial\boldsymbol{A}/\partial t = \boldsymbol{0}$, we have $\boldsymbol{E} = \boldsymbol{0}$. Some magnetic-field lines in the $z = 0$ plane, obtained from the isocontours of A_y, are shown below, with those in red revealing a magnetic bottle:



(e) Place a particle in the center of the mirror and launch it with velocity $\boldsymbol{v}$. Discuss with your group how the particle moves for various initial pitch angles, $v_x(0)/v(0)$.

Shown below are particle trajectories in the x-y and y-z planes from a numerical integration at different pitch angles $\theta_0 \doteq \cos^{-1}[v_x(0)/v(0)]$ in a double-dipole mirror with $\boldsymbol{m} = 500\hat{\boldsymbol{x}}$ and $\boldsymbol{a} = 10\hat{\boldsymbol{x}}$. The red-orange-yellow-green-blue-violet color sequence corresponds to time running from $t = 0$ to $t_\mathrm{f} = 60\pi$:





A number of effects are evident: (i) particles with larger $v_\parallel/v$ (i.e., smaller θ_0) penetrate farther into the magnetic potential before turning around – a consequence of conservation of energy and μ; (ii) particles are curvature and grad-B drifting,

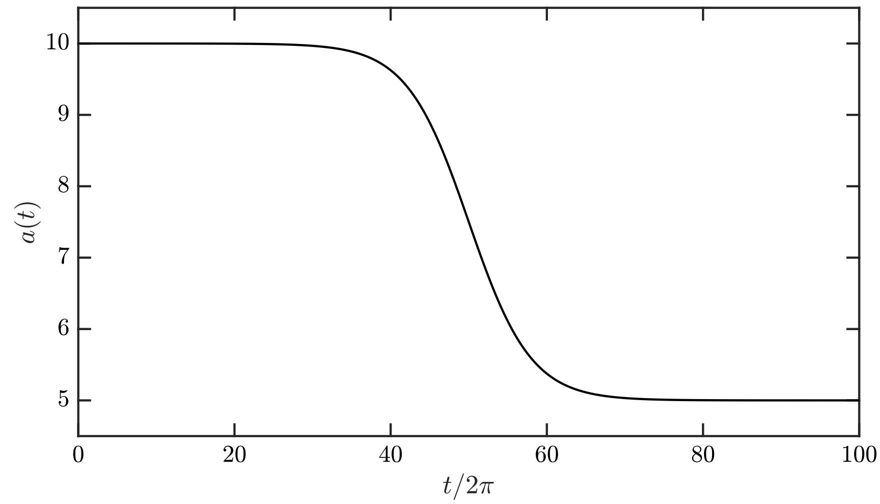
$$\boldsymbol{v}_{\rm curv} + \boldsymbol{v}_{\nabla B} = \frac{v^2}{\Omega}\left(1+\cos^2\theta\right)\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\ln B,$$

in the $-\hat{\boldsymbol{\varphi}}$ direction; and (iii) smaller θ_0 have larger gyro-radii and thus larger drifts.

(f) Suppose the distance between the two dipoles in part (e) is adiabatically shrunk in half:

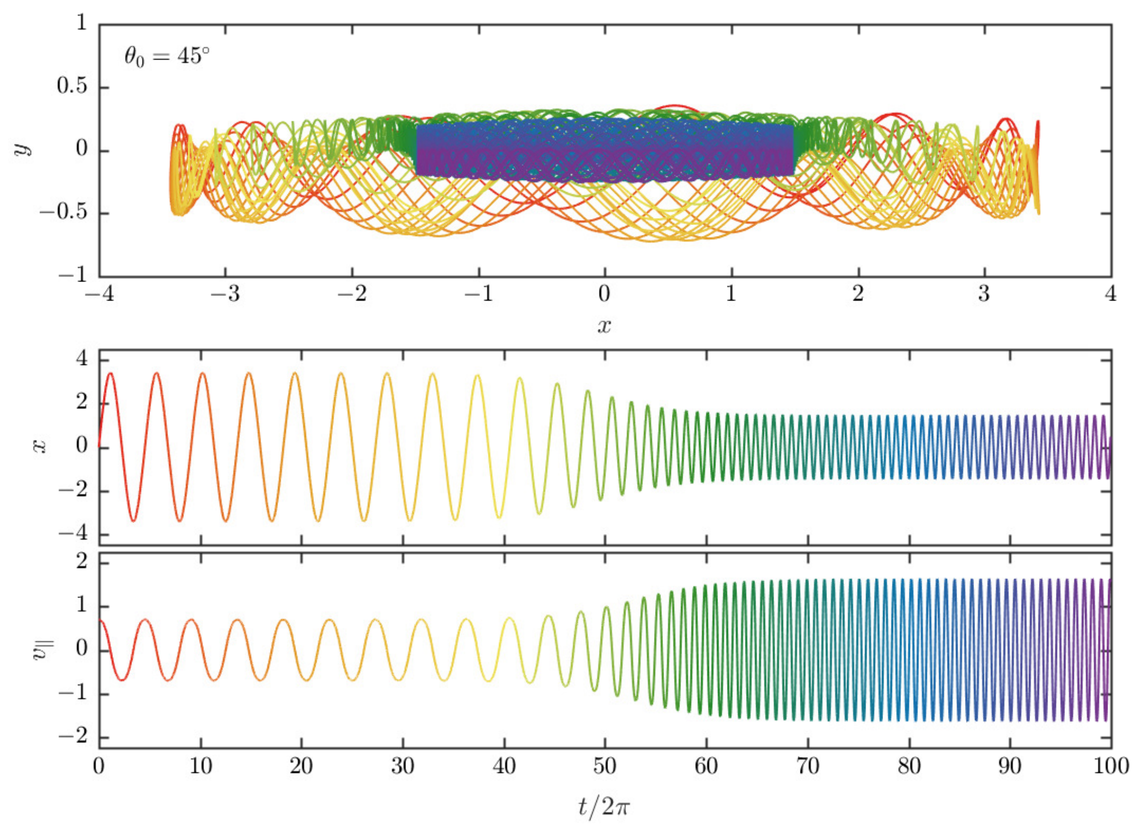
$$\boldsymbol{a} \rightarrow \boldsymbol{a}(t) = 10\hat{\boldsymbol{x}} - 2.5\big\{1+\tanh[\gamma(t-t_{\rm f}/2)]\big\}\hat{\boldsymbol{x}},$$

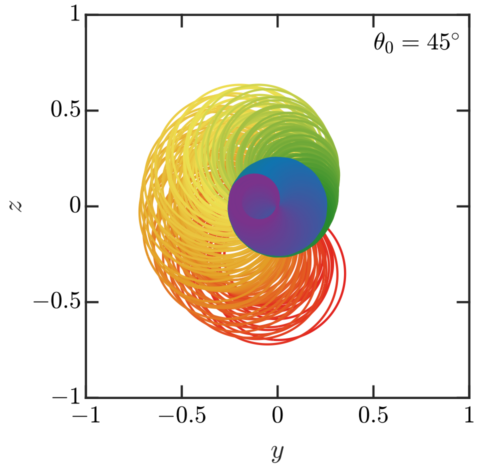
with $\gamma \ll 1$, as shown in the figure below:



The vector potential defined by equation (17) then depends upon time, $\boldsymbol{A}(\boldsymbol{r}) \to \boldsymbol{A}(t, \boldsymbol{r})$, and so there is a non-zero electric field, $\boldsymbol{E}(t, \boldsymbol{r}) = -\partial \boldsymbol{A}/\partial t$. Discuss with your group how the particle will move if $\boldsymbol{v}(0) = (\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})/\sqrt{2}$ (i.e., an initial pitch angle of 45°). In particular, what will $v_\| = \boldsymbol{v} \cdot \hat{\boldsymbol{b}}$ look like versus time?

Adopting $\boldsymbol{m} = 500\hat{\boldsymbol{x}}$, the trajectory of a $\theta_0 = 45°$ particle in the x-y and y-z planes, as well as x and $v_\|$ versus time, are shown below from a numerical integration:





Note that, as $\boldsymbol{a}$ shrinks, the turning points contract and, by conservation of J, the extrema of $v_\parallel$ increase in magnitude. An approximate check on J conservation can be had by measuring the factor by which the x-distance between the turning points is reduced ($6.8589/2.9556 = 2.3206$) and comparing it to the factor by which $|v_\parallel|$ increases ($1.6465/0.7071 = 2.3285$). Not too bad! A more accurate calculation would involve the path integral of $v_\parallel$ along the guiding-center orbit evaluated between the turning points.

8. **Critical balance.** In a rigidly rotating, hydrodynamic, incompressible fluid, the characteristic linear frequency of waves is $\omega = \pm(k_\parallel/k)\Omega$, where $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$ is the angular velocity of the flow and $k_\parallel = k_z$ is component the wavenumber oriented parallel to the rotation axis. Suppose that such a fluid is turbulent, with velocity fluctuations satisfying $k_\parallel/k_\perp \ll 1$, i.e., the fluctuations are anisotropic with respect to the rotation axis and elongated in that direction. Assume the turbulence to be strong and critically balanced. Obtain the resulting perpendicular and parallel power spectra of the turbulent velocities and the scaling relation linking $k_\parallel$ and $k_\perp$. Does the anisotropy of the fluctuations increase or decrease as the cascade goes to smaller scales? Is the similar to or different than Goldreich–Sridhar turbulence?

Following standard Kolmogorov arguments for the perpendicular cascade,

$$\varepsilon \sim \frac{\delta u_\lambda^2}{\tau_\lambda} \sim \frac{\delta u_\lambda^3}{\lambda} \quad \Longrightarrow \quad E(k_\perp) \sim \varepsilon^{2/3} k_\perp^{-5/3},$$

where λ denotes the perpendicular scale. To obtain the scalings along the rotation axis, we first note that the linear frequency $\omega = (k_\parallel/k)\Omega \approx (k_\parallel/k_\perp)\Omega$ if the turbulence is strongly anisotropic. Denoting the parallel scale by ℓ, the critical balance is

$$\Omega\,\frac{\lambda}{\ell} \sim \frac{\delta u_\lambda}{\lambda} \sim \varepsilon^{1/3}\lambda^{-2/3} \quad \Longrightarrow \quad \ell \sim \Omega\,\varepsilon^{-1/3}\lambda^{5/3}.$$

In terms of wavenumbers, $k_\parallel \sim \Omega^{-1}\varepsilon^{1/3}k_\perp^{5/3}$. Note that $k_\parallel/k_\perp \propto k_\perp^{2/3}$, and so the anisotropy decreases as the cascade goes to smaller scales. At $k_\parallel \sim k_\perp \sim k_{\rm iso} \sim \Omega^{3/2}\epsilon^{-1/2}$, the cascade becomes isotropic and the presence of rotation no longer matters. This is different from G–S turbulence, in which the fluctuations become more anisotropic with respect to the magnetic field as they cascade. The parallel spectrum is $E(k_\parallel) = E(k_\perp)(\mathrm{d}k_\perp/\mathrm{d}k_\parallel) \propto k_\parallel^{-7/5}$.

9. **Landau damping via Newton's 2nd**. Imagine an electron moving along the z axis with constant speed v_0. Slowly turn on a wave-like electric field: $\boldsymbol{E}(t,z) = E_0\cos(\omega t - kz)\,\mathrm{e}^{\epsilon t}\hat{\boldsymbol{z}}$, where ω is the frequency and k is the wavenumber of the wave; the adverb "slowly" is captured by the $\mathrm{e}^{\epsilon t}$ factor with $\epsilon \ll 1$. (You'll take $\epsilon \to +0$ at the end of the calculation.) The goal is to solve this problem perturbatively by assuming E_0 is so small that it changes the electron's trajectory only a little bit over several wave periods.

(a) The lowest-order solution is $v_z(t) = v_0$, $z(t) = v_0 t$, and $E(t, v_0 t) = E_0\cos[(\omega - kv_0)t]\,\mathrm{e}^{\epsilon t}$. Calculate the first-order corrections, $\delta v_z(t)$, $\delta z(t)$, and $\delta E(t,z)$.

The equations of motion are

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v_z \quad \text{and} \quad \frac{\mathrm{d}v_z}{\mathrm{d}t} = -\frac{e}{m_\mathrm{e}}E_0\cos(\omega t - kz)\,\mathrm{e}^{\epsilon t}.$$

The lowest-order solution is simple: $z(t) = v_0 t$ and $v_z(t) = v_0 = \text{const}$. Write $v_z(t) = v_0 + \delta v_z(t)$ and $z(t) = v_0 t + \delta z(t)$. Then,

$$\frac{\mathrm{d}\delta v_z}{\mathrm{d}t} = -\frac{e}{m_\mathrm{e}}E(t, z(t)) \approx -\frac{e}{m_\mathrm{e}}E(t, v_0 t) = -\frac{e}{m_\mathrm{e}}E_0\,\Re\left\{\mathrm{e}^{[\mathrm{i}(\omega - kv_0)+\epsilon]t}\right\}$$

$$\begin{aligned}
\delta v_z(t) &= -\frac{eE_0}{m_\mathrm{e}}\int_0^t \mathrm{d}t'\,\Re\left\{\mathrm{e}^{[\mathrm{i}(\omega-kv_0)+\epsilon]t'}\right\} \\
&= -\frac{eE_0}{m_\mathrm{e}}\Re\left\{\frac{\mathrm{e}^{[\mathrm{i}(\omega-kv_0)+\epsilon]t} - 1}{\mathrm{i}(\omega - kv_0) + \epsilon}\right\} \\
&= -\frac{eE_0}{m_\mathrm{e}}\frac{\epsilon\mathrm{e}^{\epsilon t}\cos[(\omega - kv_0)t] - \epsilon + (\omega - kv_0)\mathrm{e}^{\epsilon t}\sin[(\omega - kv_0)t]}{(\omega - kv_0)^2 + \epsilon^2} \\
\delta z(t) &= \int_0^t \mathrm{d}t'\,\delta v_z(t') \\
&= -\frac{eE_0}{m_\mathrm{e}}\int_0^t \mathrm{d}t'\,\Re\left\{\frac{\mathrm{e}^{[\mathrm{i}(\omega-kv_0)+\epsilon]t} - 1}{\mathrm{i}(\omega - kv_0) + \epsilon}\right\} \\
&= -\frac{eE_0}{m_\mathrm{e}}\left[\Re\left\{\frac{\mathrm{e}^{[\mathrm{i}(\omega-kv_0)+\epsilon]t} - 1}{[\mathrm{i}(\omega - kv_0) + \epsilon]^2}\right\} - \frac{\epsilon t}{(\omega - kv_0)^2 + \epsilon^2}\right] \\
&= -\frac{eE_0}{m_\mathrm{e}}\left\{\frac{[\epsilon^2 - (\omega - kv_0)^2][\mathrm{e}^{\epsilon t}\cos[(\omega - kv_0)t] - 1] + 2\epsilon(\omega - kv_0)\mathrm{e}^{\epsilon t}\sin[(\omega - kv_0)t]}{[(\omega - kv_0)^2 + \epsilon]^2}\right. \\
&\qquad\left. - \frac{\epsilon t}{(\omega - kv_0)^2 + \epsilon^2}\right\}.
\end{aligned}$$

$$\delta E(t,z) = E(t,z) - E(t, v_0 t) = \delta z(t)\frac{\partial E}{\partial z}(t, v_0 t) = \delta z(t) k\sin[(\omega - kv_0)t]E_0\mathrm{e}^{\epsilon t}.$$

(b) The average power gained by the electron (and thus lost by the wave) is

$$P(v_0) = -e\langle E(t, z(t))v_z(t)\rangle \approx -e\langle [E(t, v_0 t) + \delta E(t,z)][v_0 + \delta v_z(t)]\rangle,$$

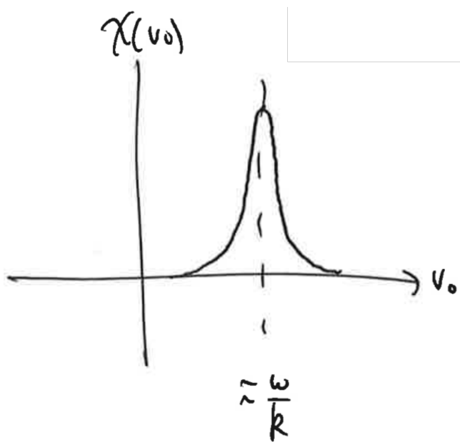
where the brackets indicate an average over timescales satisfying $\omega^{-1} \ll t \ll \epsilon^{-1}$. Use this to show that, to leading order,

$$P(v_0) = \frac{e^2 E_0^2}{2m_\mathrm{e}}\,\mathrm{e}^{2\epsilon t}\,\frac{\mathrm{d}\chi}{\mathrm{d}v_0}, \quad \text{where} \quad \chi(v_0) \doteq \frac{\epsilon v_0}{(\omega - k v_0)^2 + \epsilon^2}. \tag{18}$$

Plot $\chi(v_0)$ and identify when $P(v_0) > 0$ and $P(v_0) < 0$. Explain what each case means physically.

$$\begin{aligned}
P(v_0) &= -e\langle E(t, z(t)) v_z(t)\rangle \\
&\approx -e\langle [E(t, v_0 t) + \delta E(t, z)][v_0 + \delta v_z(t)]\rangle \\
&= -e\Big\langle \underbrace{v_0 E(t, v_0 t)}_{=0\text{ after averaging}} + \underbrace{\delta v_z(t) E(t, v_0 t)}_{\substack{\text{only cos term}\\ \text{in } \delta v_z \text{ survives}}} + \underbrace{\delta E(t, z) v_0}_{\substack{\text{only sin term}\\ \text{in } \delta z \text{ survives}}}\Big\rangle + \mathcal{O}(\delta^2) \\
&= -eE_0\,\mathrm{e}^{\epsilon t}\Big\langle -\frac{eE_0}{m_\mathrm{e}}\frac{\epsilon \mathrm{e}^{\epsilon t}}{(\omega - k v_0)^2 + \epsilon^2}\cos^2[(\omega - k v_0)t] \\
&\qquad - \frac{eE_0}{m_\mathrm{e}}\frac{2\epsilon k v_0(\omega - k v_0)\mathrm{e}^{\epsilon t}}{[(\omega - k v_0)^2 + \epsilon^2]^2}\sin^2[(\omega - k v_0)t]\Big\rangle \\
&= \frac{e^2 E_0^2}{2m_\mathrm{e}}\,\mathrm{e}^{2\epsilon t}\left[\frac{\epsilon}{(\omega - k v_0)^2 + \epsilon^2} + \frac{2\epsilon k v_0(\omega - k v_0)}{[(\omega - k v_0)^2 + \epsilon^2]^2}\right] \\
&= \frac{e^2 E_0^2}{2m_\mathrm{e}}\,\mathrm{e}^{2\epsilon t}\frac{\mathrm{d}}{\mathrm{d}v_0}\underbrace{\left[\frac{\epsilon v_0}{(\omega - k v_0)^2 + \epsilon^2}\right]}_{\doteq\,\chi(v_0)}
\end{aligned}$$

Plot of $\chi(v_0)$ and explanation:



- if $v_0 \lesssim \omega/k$ (particle lagging), then $\frac{d\chi}{dv_0} > 0 \Rightarrow P(v_0) > 0 \Rightarrow$ energy goes from field to e^- (wave is damped)
- if $v_0 \gtrsim (\omega/k)$ (particle leading), then $\frac{d\chi}{dv_0} < 0 \Rightarrow P(v_0) < 0 \Rightarrow$ energy goes from e^- to field (wave grows)

(c) This must be a very lonely electron, so let's give him some friends. Suppose there is now a whole distribution of these electrons, $F(v_0)$. Show that the total power per unit volume going into (or out of) this distribution is (take $\epsilon \to +0$)

$$P = -\frac{e^2 E_0^2}{2m_\mathrm{e} k^2} \pi \omega F'\left(\frac{\omega}{k}\right). \tag{19}$$

Explain this formula in the context of Landau damping. You'll need Plemelj's formula:

$$\lim_{\epsilon \to +0} \frac{1}{x - \zeta \mp \mathrm{i}\epsilon} = \mathrm{PV}\frac{1}{x - \zeta} \pm \mathrm{i}\pi\delta(x - \zeta),$$

where PV denotes the principal value and $\delta(x)$ is the Dirac delta function.

$$\begin{aligned} P = \int \mathrm{d}v_z\, F(v_z) P(v_z) &= \frac{e^2 E_0^2}{2m_\mathrm{e}} \mathrm{e}^{2\epsilon t} \int \mathrm{d}v_z\, F(v_z) \frac{\mathrm{d}\chi}{\mathrm{d}v_z} \\ &= -\frac{e^2 E_0^2}{2m_\mathrm{e}} \mathrm{e}^{2\epsilon t} \int \mathrm{d}v_z\, F'(v_z) \chi(v_z). \end{aligned}$$

Take $\epsilon \to 0^+$:

$$\begin{aligned} \chi(v_z) &= \frac{\epsilon v_z}{(\omega - k v_z)^2 + \epsilon^2} \\ &= -\frac{\mathrm{i} v_z}{2}\left(\frac{1}{k v_z - \omega - \mathrm{i}\epsilon} - \frac{1}{k v_z - \omega + \mathrm{i}\epsilon}\right) \\ &\to \frac{\pi \omega}{k^2}\, \delta(v_z - \omega/k) \\ \implies \quad P &= -\frac{e^2 E_0^2}{2m_\mathrm{e}} \frac{\pi\omega}{k^2} F'\left(\frac{\omega}{k}\right). \end{aligned}$$

Damping if $\omega F'(\omega/k) < 0$. Instability if $\omega F'(\omega/k) > 0$. Makes sense!