SHOCKS

SHOCK WAVE : PRESSURE DISTURBANCE FASTER THAN SIGNAL SPEED FOR COMPRESSIVE WAVES

WHERE ?

- STELLAR EXPLOSIONS
- FAST STELLAR (SOLAR) WINDS
- BULLET / PLANE
- ...

PLAN :
- RECAP OF HYDRO EQS
- WAVE STEEPENING INTO SHOCKS
- SHOCK JUMP CONDITIONS
- FUN FACTS ABOUT SHOCKS

HYDRO EQS

MASS CONSERVATION EQ.
$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho \vec{v}) = 0}$$

MOMENTUM EQ. (EULER)
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,\vec{v} = -\frac{\nabla P}{\rho}$$

ASSUME : ISOTROPIC PRESSURE

NO VISCOSITY (IDEAL FLUID)

NO EXTERNAL FORCES

INTERNAL ENERGY PER UNIT MASS

ENERGY EQ:
$$\rho \frac{d\varepsilon}{dt} + P\,\vec{\nabla}\cdot\vec{v} = (-\mathcal{L}) = 0$$

COMPRESSION / EXPANSION

$\mathcal{L}$: HEAT LOSS RATE PER UNIT VOLUME — RADIATION, CONDUCTION

$\mathcal{L} = 0 \Leftrightarrow$ ADIABATIC

CAREFUL : $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})$

CAST THEM IN <u>CONSERVATIVE</u> FORM

$$\frac{\partial}{\partial t}(Q) + \vec{\nabla}\cdot(\text{FLUX OF } Q) = 0$$

(OR EXTERNAL FORCES / HEAT GAINS OR LOSSES)

$\boxed{\text{MASS}}$: ALREADY CONSERVATIVE $\rho\vec{v}$ IS MASS FLUX

$\boxed{\text{MOMENTUM}}$ $$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(R_{ij}) = 0$$

R_{ij} : FLUX OF MOM IN DIRECTION i ACROSS SURFACE WHOSE OUTWARD NORMAL IS j

$\boxed{\text{REYNOLDS STRESS TENSOR}}$

$$R_{ij} = \underbrace{\rho v_i v_j}_{\substack{\text{BULK} \\ \text{(RAM PRESSURE)}}} + \underbrace{P\,\delta_{ij}}_{\substack{\text{THERMAL} \\ \text{(INTERNAL)}}}$$

NOT ε! ENERGY CHANGE ALSO DUE TO ADIABATIC COMPRESSION/EXPANSION

$\boxed{\text{ENERGY}}$ $$\frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2}+\varepsilon\right)\right] + \vec{\nabla}\cdot\left[\rho\vec{v}\left(\frac{v^2}{2}+W\right)\right] = 0$$

$$W = \varepsilon + \frac{P}{\rho}$$ ENTHALPY PER UNIT MASS

WHY DID WE DO THIS?
- NUMERICALLY USEFUL
- USEFUL FOR SHOCKS (SEE LATER)

WAVE STEEPENING INTO SHOCKS

BY LINEARIZING HYDRO EQS YOU DERIVED SOUND WAVES

$$c_s^2 \equiv \left.\frac{\partial P}{\partial \rho}\right|_s$$ SOUND SPEED (ADIABATIC)

(IF EVOLUTION IS FAST COMPARED TO NON-ADIAB. PROCESSES)

IN MOM. EQUATION, YOU NEGLECTED $(\vec{v}_1 \cdot \vec{\nabla})\vec{v}_1$

$\vec{v}_0 = 0$ UNPERTURBED

$\vec{v}_1 \neq 0$ PERTURBED BUT SMALL $\rightarrow$ NEGLECT 2^{nd} ORDER TERMS

WHAT IF $\vec{v}_1$ IS NOT SMALL ?

CONSIDER MOM EQ IN 1d
$$\frac{\partial v}{\partial t} + \underbrace{v \frac{\partial v}{\partial x}}_{\sim \frac{v^2}{L}} = \underbrace{-\frac{1}{\rho} \frac{\partial p}{\partial x}}_{\sim \frac{c_s^2}{L}}$$

IF $v^2 >> c_s^2$ RHS CAN BE NEGLECTED

$$\rightarrow \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0$$

NOW CONSIDER $\frac{dx}{dt} = v$ (CURVE IN (x,t) PLANE)

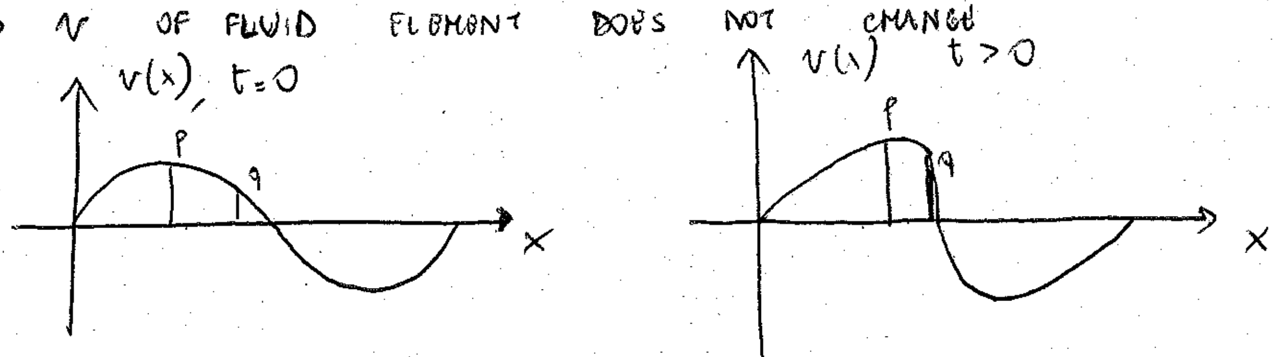
ALONG THIS CURVE
$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \underbrace{v}_{\frac{dx}{dt} = v} \frac{\partial v}{\partial x} = 0 \rightarrow \boxed{\frac{dv}{dt} = 0}$$

FROM PDE TO ODE ON SPECIFIC CURVES (METHOD OF CHARACTERISTICS)

FOR US; v = CONST ON CURVE IN (x,t) PLANE

(CHARACTERISTIC CURVE HERE IS LAGRANGIAN TRACK OF FLUID ELEMENT)

$\Rightarrow$ v OF FLUID ELEMENT DOES NOT CHANGE



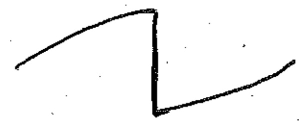
$v(p) > v(q)$ SO THEY MOVE CLOSER

EVENTUALLY ....

EVENTUALLY OVERTAKES 

UNPHYSICAL! ← MULTI-VALUED

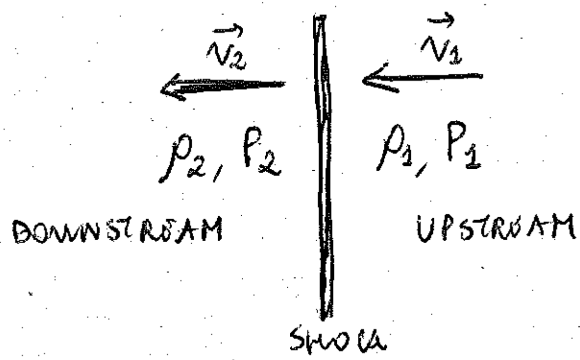
BOTTOM LINE: $v \frac{\partial v}{\partial x}$ STEEPENS, AT SOME POINT RHS KICKS IN

→ SHOCK! 

$\boxed{\text{SHOCKS}}$ ANY NON-INFINITESIMAL PERTURBATION WITH $v > c_s$ DEVELOPS A SHOCK!

MATH: DISCONTINUITY OF HYDRO QUANTITIES

PHYSICS: JUMP ON SCALES $\ll$ SIZE OF SYSTEM



CONSERVATIVE EQS.: $\frac{\partial Q}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

↳ FLUX OF Q

IN SHOCK FRAME, STEADY STATE → $\frac{\partial Q}{\partial t} = 0 \Rightarrow \frac{\partial J}{\partial x} = 0 \Rightarrow J_2 = J_1$

MASS: $[\rho v_x] = 0$ → MEANING: $\rho_1 v_{1x} = \rho_2 v_{2x}$

MOMENTUM: $[\rho v_x^2 + P] = 0$ $\quad [\rho v_x v_y] = [\rho v_x v_z] = 0$

ENERGY: $[\rho v_x (\frac{1}{2} v^2 + w)] = 0$

TWO OPTIONS: (A) ZERO MASS FLUX: TANGENTIAL DISCONTINUITY

$v_{1x} = v_{2x} = 0$

$\Rightarrow P_1 = P_2$

v_{1t} v_{2t} ARBITRARY (IF ZERO, CONTACT DISCONTINUITY)

↓

TANGENTIAL

(B) SHOCK

$$[\rho v_x] = 0$$
$$\left[\frac{1}{2} v^2 + w\right] = 0$$
$$[v_y] = [v_z] = 0 \rightarrow \text{MOVE INTO FRAME WHERE THEY ARE ZERO}$$
$$[P + \rho v_x^2] = 0$$

$\Downarrow$

$v = v_x$

LET'S COUNT: ρ_2, v_2, P_2, w_2

$$w_2 = \varepsilon_2 + \frac{P_2}{\rho_2} = \frac{\gamma P_2}{(\gamma - 1)\rho_2} \qquad \gamma = \text{ADIABATIC INDEX}$$

DEFINE $M_1 = \dfrac{v_1}{c_{s,1}} = \dfrac{v_1}{\sqrt{\gamma \frac{P_1}{\rho_1}}}$ MACH NUMBER, FOR SHOCKS > 1

& SOLVE!

"JUMP" CONDITIONS

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{\gamma + 1}{2/M_1^2 + (\gamma - 1)} > 1$$

FOR $M_1 \to \infty$ $\quad \dfrac{\rho_2}{\rho_1} = \dfrac{\gamma + 1}{\gamma - 1} \quad \left(= 4 \text{ IF } \gamma = \frac{5}{3}\right)$

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1} \Rightarrow T_{2, M_1 \to \infty} = \frac{2(\gamma - 1)}{(\gamma + 1)} m v_1^2$$

KINETIC BULK ENERGY TRANSFORMED TO HEAT!

FUN FACTS

1) $M_1 > 1, \quad M_2 = \dfrac{v_2}{c_{s,2}} < 1$ SUPERSONIC PLANES!

2) $s_2 > s_1$ (ENTROPY PER PARTICLE) $\iff M_1 > 1$

3) IF $M_1 < 1$ & WE TAKE SHOCK JUMP CONDITIONS

$\rightarrow v_1 < v_2, \quad T_2 < T_1$

$\rightarrow$ TRANSFORM HEAT INTO ORDERED ENERGY / WORK

$\rightarrow$ VIOLATES 2^{nd} LAW OF THERMODYNAMICS

MHD SHOCKS

WE CAN ALSO WRITE MHD EQS IN CONSERVATIVE FORM

MOMENTUM $$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Big[P\delta_{ij} + \rho v_i v_j - \underbrace{\frac{1}{4\pi}\Big(B_i B_j - \frac{B^2}{2}\delta_{ij}\Big)}_{\text{MAXWELL STRESS TENSOR}}\Big] = 0$$

MHD SHOCKS NEED TO BE SUPER SONIC & SUPER ALFVENIC $\quad v_A = \frac{B}{\sqrt{4\pi\rho}}$

(SUPER-FAST-MAGNETOSONIC)

IN MOST CASES $\beta = \frac{P_{gas}}{P_{mag}} \sim \frac{c_s^2}{V_A^2} \gtrsim 1$ SO SUPERSONIC OFTEN IS ENOUGH

JUMP CONDITIONS : - MASS, MOMENTUM, ENERGY CONS.

- B_n, E_t CONTINUOUS $\quad \Big\langle\ \vec{\nabla}\cdot\vec{B} = 0,\quad -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\wedge\vec{E}$

$n \leftrightarrow$ NORMAL

$t \leftrightarrow$ TANGENTIAL

$$\Rightarrow [B_n] = 0 \quad (6)$$

$$[E_t] = 0 \quad , \quad \vec{E} = -\frac{\vec{v}}{c}\wedge\vec{B}$$

$$\Rightarrow [B_n v_t - B_t v_n] = 0 \quad (5)$$

IN ADDITION,

(1) $[\rho v_n] = 0$ MASS

(2) $\Big[\rho v_n v_t - \frac{1}{4\pi} B_n B_t\Big] = 0$ MOMENTUM $\perp$ SHOCK NORMAL

(3) $\Big[P + \rho v_n^2 + \frac{1}{8\pi}(B_t^2 - B_n^2)\Big] = 0$ MOMENTUM //

(4) $\Big[\rho v_n\Big(\frac{v^2}{2} + w\Big) + \frac{1}{4\pi}\big(v_n B^2 - B_n(\vec{v}\cdot\vec{B})\big)\Big] = 0$

ENERGY

TWO SIMPLE CASES (TAKE $V_t = 0$ BY FRAME TRANSFORM)

(A) $B_{t,1} = 0$ AHEAD $\rightarrow B_{t,2} = 0$ "PARALLEL" SHOCK

B_n DECOUPLES, WE ARE BACK TO HYDRO EQS

(B) $B_{n,1} = 0$ "PERPENDICULAR" SHOCK

$$\frac{B_{t2}}{B_{t1}} = \frac{\rho_2}{\rho_1} = \frac{v_{1n}}{v_{2n}}$$ FLUX FREEZING (FROM $[E_t] = 0$)

AND THEN $[\rho v] = 0$ $\left[P + \rho v^2 + \frac{B_t^2}{8\pi} \right] = 0$ MOM //

$$\left[\frac{1}{2} v^2 + w + \frac{B_t^2}{4\pi\rho} \right] = 0$$ ENERGY

SO, B FIELD ENTERS IN PRESSURE & ENERGY DENSITY.

SHOCK THICKNESS

VISCOSITY (NEGLECTED SO FAR) SCALES AS $\frac{\partial^2}{\partial x^2}$
$\rightarrow$ IMPORTANT IN THE STEEP SHOCK TRANSITION

IF VISCOSITY (COLLISIONS) SETS SHOCK THICKNESS $\rightarrow \Delta \sim \lambda_{mfp} = \frac{1}{n\sigma}$

MEAN FREE PATH

EARTH'S ATMOSPHERE $n \sim 3 \times 10^{19}\ cm^{-3}$

$$\sigma \sim (\overset{\text{FEW}}{\pi}\, a_B^2) \sim 3 \times 10^{-15}\ cm^2$$

BOHR RADIUS

$\rightarrow \lambda_{mfp}$ SMALL! $\sim \frac{1}{3 \times 10^{19} \cdot 3 \times 10^{-15}} \sim 10^{-5}\ cm$

FOR SN SHOCK (STELLAR EXPLOSION)

$n \sim 1\ cm^{-3}$ $\quad \sigma \sim \pi R_N^2 \sim (2 \times 10^{-13}\ cm)^2 \cdot \pi$

RADIUS OF NUCLEUS (IONIZED)

$\lambda_{mfp} \sim 10^{25}\ cm$ BUT SN RADIUS $\sim 1\,pc \sim 3 \times 10^{18}\ cm$

SHOCK NOT MEDIATED BY BINARY COLLISIONS, BUT BY EM FIELDS

→ COLLISION-$\boxed{\text{LESS}}$ SHOCKS

$$\text{THICKNESS} \sim r_L \sim \frac{m_p v c}{e B} \sim 10^{11}\ cm\ \frac{v}{10^4\ km/s}\ \frac{10^{-6}\ G}{B}$$

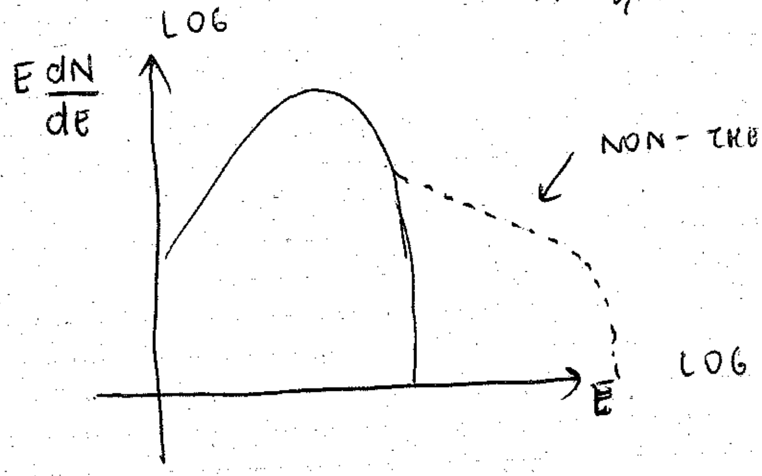
FUN FACTS ABOUT COLLISIONLESS SHOCKS

1) $T_e \neq T_i$ (e - i EQUILIBRATION TIME >> SYSTEM EVOLUTION)

2) NON-THERMAL PARTICLES

(EQUILIBRATION TIME WITHIN A GIVEN SPECIES

>> SYSTEM EVOLUTION)



POWER-LAW $\frac{dN}{dE} \propto E^{-s}$

S = POWER-LAW SLOPE

COSMIC RAYS

CHARGED PARTICLES ORIGINATING BEYOND EARTH (MOSTLY HADRONS)

HIGHEST ENERGIES $\sim 10^{21}$ eV

$1\ \mathrm{eV} \approx 1.6 \times 10^{-12}$ erg

1.6×10^{-19} J

COMPARE WITH TENNIS BALL KINETIC ENERGY

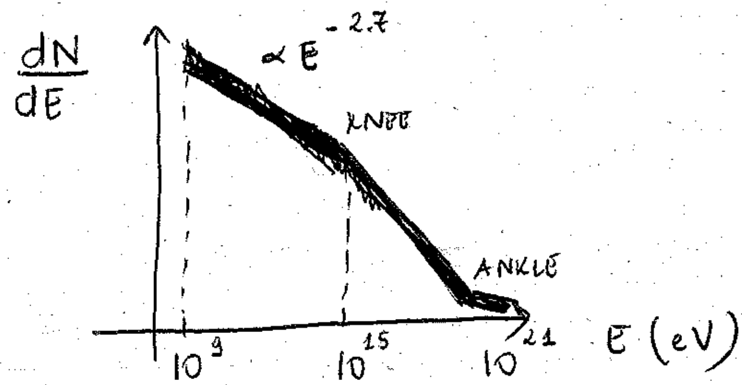
$m \sim 100\,g \sim 0.1$ Kg

$v \sim 100$ km/hr ~ 30 m/s

$\rightarrow E \sim \frac{1}{2}\, 10^{2} \cdot 10^{-3} \cdot 30^{2} \sim 50\ \mathrm{J} \lesssim 10^{21}$ eV!

CALCULATION IN MKS

WE THINK CRs UP TO THE "KNEE", $E \sim 3\times 10^{15}$ eV, ARE PRODUCED @ SUPERNOVA REMNANT SHOCKS



TWO QUESTIONS:

1) CAN SN SHOCKS ACCELERATE /CONFINE PeV PARTICLES?

2) WHAT IS THE SPECTRUM $\left(\frac{dN}{dE}\right)$ OF ACCELERATED COSMIC RAYS?

1) "HILLAS" CRITERION $r_L < L$ (SIZE OF SYSTEM)

r_L FOR 3 PeV PROTONS: $r_L \sim \frac{E}{eB} \sim \frac{3\times 10^{15} \cdot 10^{-12}}{4\times 10^{-10}\ 10^{-6}} \sim 10^{19}$ cm

$L \sim 1\,pc \sim 3 \times 10^{18}$ cm

THE EXPLOSIVE IGNORANCE

- SNR BLAST WAVE (EXPLOSION) IS EXPANDING SPHERICALLY
- ENERGY E, EXPLOSION MASS M_{eJ}, SWEEPS UP MEDIUM WITH MASS DENSITY ρ_0; E IS CONSERVED (ADIABATIC)
- WE LOOK AT THE EXPLOSION WHEN BLAST WAVE HAS SWEPT UP MUCH MORE MASS THAN M_{eJ}
- WHICH CONSTANTS ENTER?

  NON-RELATIVISTIC : $\cancel{c}$

  NON-QUANTUM : $\cancel{\hbar}$

  NEUTRAL : $\cancel{e}$

  NON-GRAVITATIONAL : $\cancel{G}$

- RADIUS OF BLAST WAVE WILL NECESSARILY BE (VS TIME)

$$R \sim (?) \left(\frac{E t^2}{\rho_0}\right)^{1/5}$$ FROM DIMENSIONAL ANALYSIS

AS LONG AS $\frac{4\pi}{3}\rho_0 R^3 \gg M_{eJ}$

THIS SEEMS INSUFFICIENT, EVEN WHEN TAKING INTO ACCOUNT THE 4 X COMPRESSION OF ISM FIELD !

SOLUTION ? PLASMA PHYSICS ! (BELL 2004, 2005)

CRs ACCELERATED AT SHOCKS SEED A PLASMA INSTABILITY ABLE TO AMPLIFY THE UPSTREAM FIELD EVEN BY 30 × !

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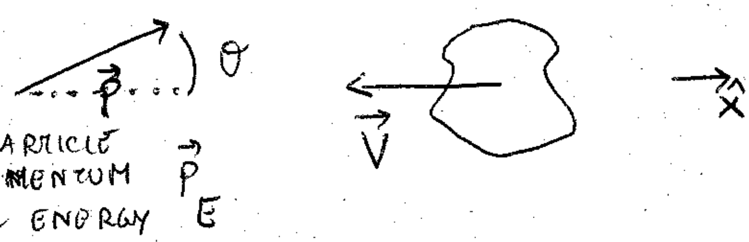
2) SHOCK ACCELERATION OF PARTICLES

BASIC IDEA (ENRICO FERMI) :

- CLOUDS MOVING AT $V \ll c$ CARRYING MAGNETIC FIELDS
- PARTICLES COLLIDE WITH CLOUDS LIKE WITH RIGID WALLS (REFLECTION IS ELASTIC IN CLOUD FRAME)



PRIMED IN CLOUD FRAME, UNPRIMED IN LAB FRAME

$$E' = \gamma_V \left(E + V p_x \right) \qquad p_x' = \gamma_V \left(p_x + \frac{VE}{c^2} \right)$$

$$\gamma_V = 1 \Big/ \sqrt{1 - \frac{V^2}{c^2}} \qquad p_x = p \cos\theta$$

$$p_x' = p' \cos\theta'$$

AFTER COLLISION, CLOUD FRAME $\quad E_1' = E'$

$$p_{x1}' = - p_x'$$

BACK TO OBSERVER FRAME (LAB) $\quad \dfrac{E_1 - E}{E} \equiv \dfrac{\Delta E}{E} \simeq \dfrac{2V\cos\theta}{c} + 2\dfrac{V^2}{c^2}$

@ 2^{nd} ORDER IN $\frac{V}{c}$

WE ASSUMED RELATIVISTIC PARTICLES

OVER TIME, MANY COLLISIONS ⇒ NEED TO AVERAGE OVER θ

AVERAGE IS NON ZERO, HEAD-ON COLLISIONS ARE MORE FREQUENT!

PROBABILITY $f(\theta)\,d\theta = \frac{1}{2}\left(1 + \frac{V\cos\theta}{c}\right)\sin\theta\, d\theta$

↑ FLUX FACTOR

$$\Rightarrow \quad \frac{\langle \Delta E \rangle}{E} = \frac{8}{3}\frac{V^2}{c^2}$$

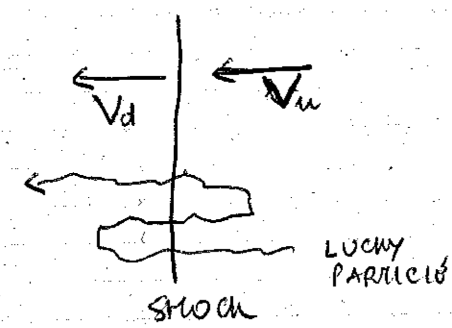
FERMI II
(2nd ORDER IN $\frac{V}{c}$) ↳ SO, KIND OF SLOW

ASTRO APPLICATIONS: TURBULENCE-DRIVEN ACCELERATION
(TURBULENT EDDIES ⟷ CLOUDS)

QUESTION: ENERGY IS INCREASING, WHERE DOES THE ENERGY COME FROM?

$\boxed{\text{SHOCK ACCELERATION}}$ (FERMI I)

SHOCK UPSTREAM & DOWNSTREAM MOVE TOWARDS EACH OTHER, SO THEY ACT AS CONVERGING CLOUDS ALWAYS



$$\frac{V_d}{V_u} = \frac{1}{r} = \frac{\rho_u}{\rho_d}$$

r = DENSITY COMPRESSION

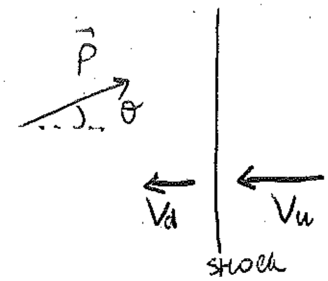
RELATIVE SPEED

$$V = V_u - V_d = \frac{r-1}{r} V_u$$

IF THE PARTICLES GET EFFICIENTLY ISOTROPIZED IN ONE REGION, THEY WILL ENCOUNTER THE OTHER AS HEAD-ON COLLISIONS → SYSTEMATIC ENERGY GAIN

(VS FERMI II, WHERE TAIL-ON COLLISIONS LED TO ENERGY LOSS)

PROBABILITY OF CROSSING?



$$f(\theta)\,d\theta = 2\underbrace{\cos\theta}_{\substack{\downarrow \\ \text{FLUX FACTOR}}}\ \underbrace{\sin\theta\, d\theta}_{\text{FOR ISOTROPIC PARTICLES}}$$

& $\cos\theta \geq 0$

FRAME TRANSFORM $D \to U \to D$ & ASSUME EFFICIENT ISOTROPIZATION

$$\left.\frac{\langle \Delta E \rangle}{E}\right|_{D \to U \to D} = \frac{4}{3}\frac{V}{c} \qquad \text{FIRST ORDER!}$$

HOW DOES THIS LEAD TO A POWER-LAW $\frac{dN}{dE} \propto E^{-s}$?

AFTER K ACCELERATION CICLES

A) $E = A^K E_0 \qquad A = 1 + \frac{4}{3}\frac{V}{c}$

B) IF P IS PROBABILITY TO REMAIN IN ACCELERATION REGION

$$N = N_0 P^k \quad (< N_0)$$

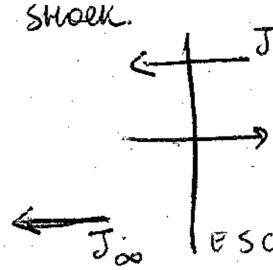
$$\underset{(\text{CUMULATIVE})}{N(\geq E)} = N_0\, P^{\ln(E/E_0)/\ln A} = N_0 \left(\frac{E}{E_0}\right)^{\frac{\ln P}{\ln A}}$$

$$\Rightarrow \underset{(\text{DIFFERENTIAL})}{\frac{dN}{dE}} \propto E^{-s} \qquad s = 1 - \frac{\ln P}{\ln A}$$

HOW TO COMPUTE P?

PARTICLES CAN ESCAPE FROM DOWNSTREAM WHERE V_d POINTS AWAY FROM THE SHOCK.

FLUXES:



J_+ ENTERING D

J_- RETURNING TO U

J_∞ ESCAPING TO D INFINITY

$J_+ = J_- + J_\infty$ (STEADY STATE) $\qquad P = \frac{J_-}{J_+} = \frac{J_-}{J_- + J_\infty}$ RETENTION PROB.

$$J_- = \int_{\cos\theta > 0} \frac{d\Omega}{4\pi}\, \underbrace{nc\cos\theta}_{\text{FLUX}} = \frac{nc}{4}$$

$J_\infty = n\, V_d$ (CARRIED AWAY AT V_d)

$$\Rightarrow P \sim 1 - \frac{4 V_d}{c}$$

$$\Rightarrow s = 1 - \frac{\ln P}{\ln A} \simeq 1 + 3\frac{V_d}{V} = 1 + \frac{3}{r-1}$$

STRONG SHOCKS HAVE $r = 4 \Rightarrow s = 2$ UNIVERSAL!

FERMI I IS FAST & LEADS TO POWER-LAW PARTICLES WITH $s = 2$