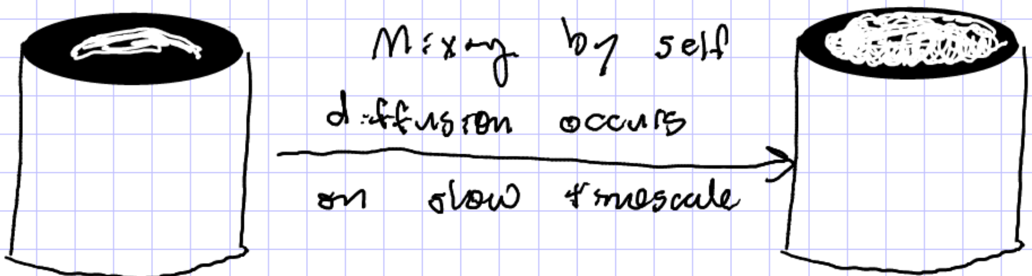
I Introduction to turbulence and K41

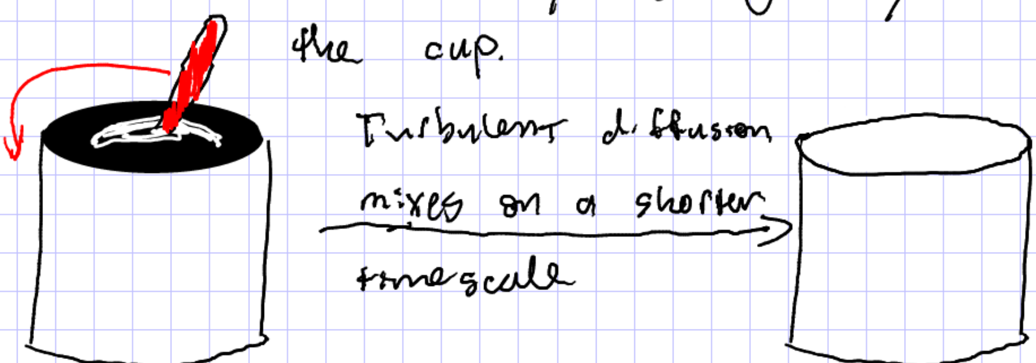
Turbulence is a chaotic flow regime characterized by diffusivity, rotationality, and dissipation.

Cartoon picture of Turbulence

Mixing of cream (tracer) into coffee



Instead, we stir. Thereby injecting energy at the scale of the cup.



Equations of Hydrodynamic Turbulence (Euler Eqns)

1) Incompressibility, $\rho = \text{const} \Rightarrow \nabla\cdot\vec{u} = 0$
2) Navier-Stokes (NS)

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\vec{u} + \vec{f}$$

Convection (Nonlinear term) — Thermal forces — Viscosity — External forcing

3) Energy: $\frac{\partial E}{\partial t} + \nabla\cdot[\vec{u}(E+p)] = 0$; $E = \frac{1}{2}\rho u^2 + \rho e$, e = internal energy

Aside: when can we assume incompressibility?

1) $\Delta\rho/\rho \ll 1$: Adiabatic change in $\Delta\rho \Rightarrow \Delta p = (\partial p/\partial\rho)_s \Delta\rho$

Bernoulli's eqn $\Rightarrow \Delta p \sim \rho u^2$ and $(\partial p/\partial\rho)_s \equiv c_s^2 = \gamma p/\rho$ Ideal gas

$\therefore \Delta\rho/\rho \ll 1 \Rightarrow u/c_s = Ma \ll 1$ $Ma :=$ Mach number

2) $\partial\rho/\partial t \ll \rho \nabla\cdot\vec{u}$: $\frac{\partial\rho}{\partial t} \sim \frac{\Delta\rho}{t} \sim \frac{\rho u^2}{t c_s^2}$; $\rho\nabla\cdot\vec{u} \sim \rho u/l$

$\therefore \frac{\rho u^2}{t c_s^2} \ll \frac{\rho u}{l}$ Assuming $u \sim \frac{l}{t} \Rightarrow t \gg l/c_s \Rightarrow$ Information propagates instantaneously

Parameters of the system:

- characteristic (outer-scale) velocity, u_0 } external forcing fixes these*
- characteristic (outer-scale) length, L }
- viscosity, ν set by molecular properties

* Outer-scale is also called the integral or autocorrelation scale $L^{int} = \int_0^\infty R_{ii}(r,t)\,dr$; $R_{ii} = \frac{\langle u_i(x_i,t)\, u_i(x_i+r,t)\rangle}{\langle u_i^2\rangle}$

Let's compare 2 important terms in the NS eqn

$$\frac{\text{convection}}{\text{viscous}} \sim \frac{u_0^2/L}{\nu u_0/L^2} = \frac{u_0 L}{\nu} =: Re \quad \text{Reynold's number}$$

- When Re is "small", viscous effects dominate and the flow is linear (laminar)
- When Re is sufficiently large, the flow becomes chaotic $\rightarrow$ turbulent. How large is large enough depends on the system, but values of $10^2 - 10^4$ are typical.
- The transition between laminar and fully developed turbulence is very messy, so we will focus on <u>fully developed turbulence</u> only.

Phenomenological picture of turbulence

- At each point in the fluid, the velocity is fluctuating around its mean value $\vec{u}_0$

$$\vec{u} = \vec{u}_0 + \delta\vec{u}$$

At the outer-scale, $\delta u_0 \sim \delta u_L$. Also, we can transform away the mean flow. So, we can redefine Re in terms of fluctuating quantities

$$Re = \frac{\delta u_L L}{\nu}$$

- Let us now consider what happens to the energy in this system $E = \frac{1}{2}\int d\vec{x}\, |\vec{u}|^2$
Dotting the NS eqn with $\vec{u}$ $\Rightarrow$

$$\frac{dE}{dt} = \underbrace{\nu \int d\vec{x}\, \vec{u}\cdot\nabla^2\vec{u}}_{\text{viscous dissipation}} + \underbrace{\int d\vec{x}\, \vec{u}\cdot\vec{f}}_{\text{rate of energy injection}}$$

If our system is in a stationary state (formally, we are considering the ensemble average $\equiv \langle\ \rangle$) then

$$\frac{d\langle E\rangle}{dt} = 0 = \nu\int d\vec{x}\, \langle \vec{u}\cdot\nabla^2\vec{u}\rangle + \underbrace{\int d\vec{x}\, \langle \vec{u}\cdot\vec{f}\,\rangle}_{=:\ \underset{\text{volume}}{V}\epsilon}$$

$$\therefore\ -\nu\int d\vec{x}\, \langle u\cdot\nabla^2 u\rangle = V\epsilon$$

In a steady state, the energy input rate, ϵ, must match the dissipation rate.

- Let's now construct estimates for various quantities based on dimensional analysis

At the outer-scale, the turbulence is characterized by u_0, L. At smaller scales, we can consider the RMS velocity u_ℓ at scale ℓ. Using just the velocity, length scale, and viscosity we now construct other important quantities

- Eddy turnover time: $t_\ell \sim \frac{\ell}{u_\ell}$

  at the outer-scale $t_L \sim \frac{L}{u_0}$

- Energy injection rate: $\epsilon \sim \frac{u_0^3}{L} \sim \frac{u_0^2}{t_L}$.

Using the energy injection rate, we can re-write

$t_L \sim \epsilon^{-1/3} L^{2/3}$ ①

- Dissipation time scale: $t_\ell^{diss} \sim \frac{\ell^2}{\nu}$ ②

- Viscous scale: Equating ① and ②

$$\Rightarrow \ell_\nu \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \sim L\, Re^{-3/4} \ll L$$

Note that this scale has multiple names: viscous scale, inner scale, dissipation scale, Kolmogorov scale are all common

Similarly, the outer-scale is often called the energy containing scale or integral scale

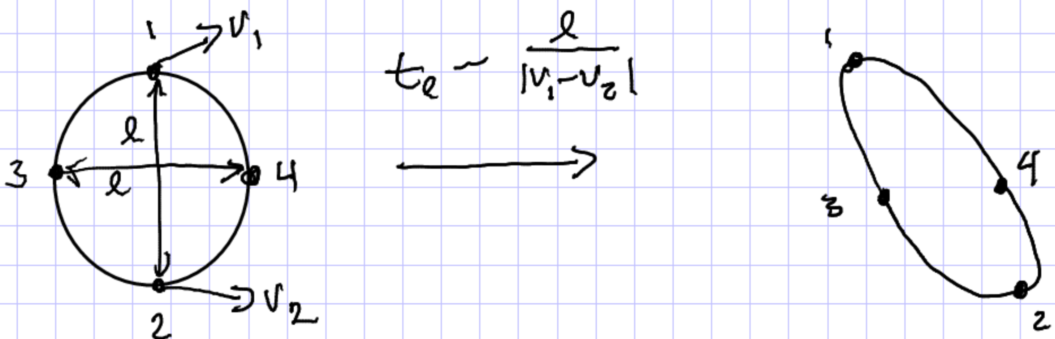
- Basic picture at this point

Energy injection → energy transport → energy dissipation

$L \gg \ell \gg \ell_\nu \sim L\, Re^{-3/4}$

Inertial Range

- Let's finish the cartoon picture of turbulence before we move on to K41.
- I defined the eddy turnover time above as $t_e \sim \frac{l}{u_l}$, but what does ~~it~~ mean? t_e is the characteristic time for a structure of size l to undergo a significant distortion due to the relative motion of its components



$$t_e \sim \frac{l}{|v_1 - v_2|}$$

$\nabla \cdot u = 0 \Rightarrow$ the area is conserved $\Rightarrow$ if 1 & 2 separate, 3 & 4 become closer. $\therefore$ t_e is also the typical time for the transfer of energy from scale l to a smaller scale. Also call this the cascade time.

Kolmogorov 1941 Turbulence Theory

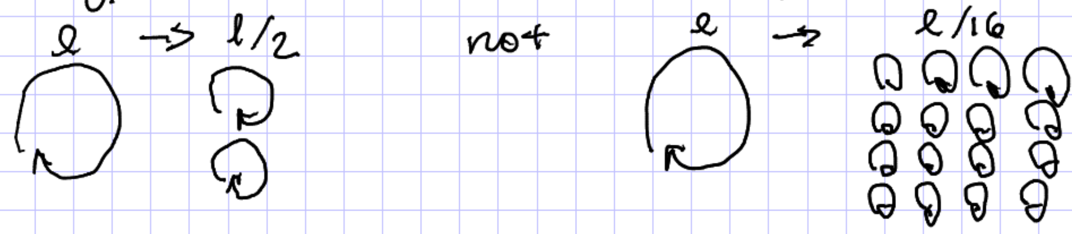
Before we discuss the theory, let's establish a baseline.

What observational facts do we know that we can use to constrain theory?

1) 2/3 law the mean square velocity increment, $\langle \delta u_l^2 \rangle$ between two points separated by l scales as $\langle \delta u_l^2 \rangle \sim l^{2/3}$

2) Finite energy dissipation the energy dissipation is always positive and finite

- In 1922, Richardson conjectured that the energy transfer is local in space to the viscous scale



but this conjecture alone does not reproduce the observable aspects of turbulence above

- So, in 1941 Kolmogorov proposed the first theory that did explain 1) & 2) above. To do so, he assumed the following

0) Universality: The turbulence, inertial range, is independent of the particular forcing (and dissipation)

1) Locality of interactions

2) Homogeneity: No special points $\Rightarrow$ no intermittency

3) Isotropy: No special directions

4) Scale invariance: No special scales $\Rightarrow$ constant cascade rate, ϵ.

- Scale invariance $\Rightarrow$ $\epsilon = \text{const}$, but we already argued that $\epsilon \sim \frac{u_0^3}{L}$, Since $\epsilon = \text{const}$,

$$\epsilon \sim \frac{\delta u_\ell^3}{\ell} \quad \therefore \quad \delta u_\ell \sim (\epsilon \ell)^{1/3} \Rightarrow \delta u_\ell^2 \sim \epsilon^{2/3} \ell^{2/3}$$

and <u>we have the $2/3$ law!</u>

$\delta u_\ell \sim (\epsilon \ell)^{1/3}$ is referred to as the Kolmogorov-Obukhov law

- Energy spectra

In general, $\vec{k} = (k_x, k_y, k_z)$, so $E^{(3)}(\vec{k})$ is the 3D energy spectrum. The total energy is thus $E = \int d\vec{k}\, E^{(3)}(\vec{k})$. If the energy is isotropic in k space ($k \in 3$), then we can use spherical coordinates

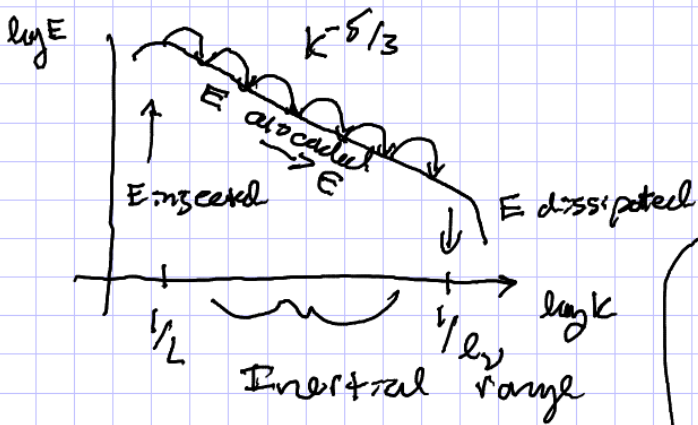
$$E = \iiint dk\, d\theta\, d\phi\, k^2 \sin\theta\, E^{(3)}(\vec{k}) = \int dk\, k^2 E^{(3)}(k) = \int dk\, E^{(1)}(k)$$

Where $E^{(1)}(k)$ is the 1D energy spectrum.

$$\therefore E = \delta u_\ell^2 \sim \epsilon^{2/3} \ell^{2/3} \sim \epsilon^{2/3} k^{-2/3} \sim \int_{k=1/\ell}^{\infty} dk\, k^2 E^{(3)} = \int dk\, E^{(1)}(k)$$

$$\Rightarrow \underbrace{E^{(1)} \sim \epsilon^{2/3} k^{-5/3}} \quad \text{and} \quad E^{(3)} \sim \epsilon^{2/3} k^{-11/3}$$

Normally use the 1D spectrum



What about scales $\ell < \ell_\nu$ and $\ell > L$?

1) $\ell > L$: $E = u_0^2 = \text{const} \Rightarrow$
$E^{(1)} \sim k^{-1}$

2) $\ell < \ell_\nu$: $\epsilon \sim \nu\, \vec{u} \cdot \nabla^2 \vec{u} \sim \nu \dfrac{\delta u_\ell^2}{\ell^2}$
$\Rightarrow \delta u_\ell \sim \left(\dfrac{\epsilon}{\nu}\right)^{1/2} \ell \Rightarrow$
$E^{(1)} \sim k^{-3}$

<u>Inertial range</u> is the range of scales unaffected by driving or dissipation. The physics is assumed to be self-similar (fractal) here.

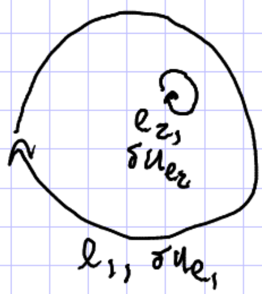
Note that $E \sim \ell^{2/3}$ dominated by large scales and gradients $\dfrac{\delta u_\ell}{\ell} \sim \ell^{-2/3}$ dominated by small scales $\Rightarrow$ viscous cutoff.

Also, the cascade time, $t_\ell \sim \dfrac{\ell}{\delta u_\ell} \sim \ell^{2/3}$ decreases with scale.

"Verifying" that the cascade is local

Consider motions at scales l_1 and l_2.

1) Can large scale motion shear apart small scales before they cascade?



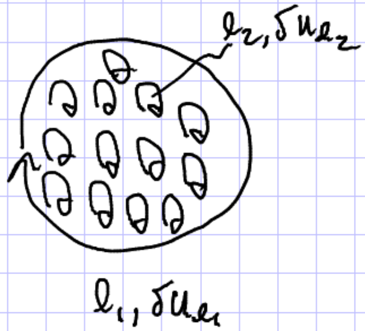
Shearing time $= t_s \sim \frac{l_1}{\delta u_{l_1}}$

cascade time for l_2 $\quad t_{l_2} \sim \frac{l_2}{\delta u_{l_2}}$

$$\frac{t_s}{t_{l_2}} \sim \frac{l_1 \delta u_{l_2}}{l_2 \delta u_{l_1}} \sim \frac{l_1 \left[\delta u_{l_1}\left(\frac{l_2}{l_1}\right)^{1/3}\right]}{l_2 \delta u_{l_1}}$$

$\sim \left(\frac{l_1}{l_2}\right)^{2/3} \gg 1 \Rightarrow$ shearing by large scales not important.

2) Can small scale eddies diffuse the large eddies before they cascade?



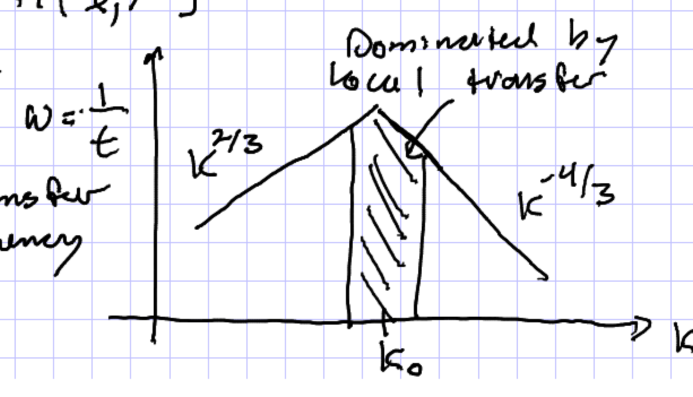
Diffusion coefficient due to eddies of size l_2: $D = \frac{l_2^2}{t_{l_2}} = l_2 \delta u_{l_2}$

Time to diffuse distance l_1 is

$$t_D \sim \frac{l_1^2}{D} \sim \frac{l_1^2}{l_2 \delta u_{l_2}}$$

$$\frac{t_D}{t_{l_1}} \sim \frac{l_1^2 \delta u_{l_1}}{\cancel{l_1} l_2 \delta u_{l_2}} \sim \frac{l_1}{l_2} \frac{\delta u_{l_1}}{\left[\delta u_{l_1}\left(\frac{l_2}{l_1}\right)^{1/3}\right]} = \left(\frac{l_1}{l_2}\right)^{4/3} \gg 1 \Rightarrow$$

diffusion is unimportant



Additional Reading

1) Uriel Frisch "Turbulence: the Legacy of A.N. Kolmogorov" 1996

2) Landau & Lifshitz "Fluid Mechanics" 1987
chapter 3

3) Kolmogorov 1941. Translation by Levin 1951
"The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers"

4) P.A. Davidson "Turbulence: An Introduction" 2004

II - Conserved Quantities and 2 vs 3D Physics

Last time, we focused on the Navier-Stokes (NS) eqn. Let's continue massaging it for more information. We will continue to assume $\nabla \cdot \vec{u} = 0$ throughout

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \nu \nabla^2 \vec{u} + \vec{f}_u \quad (1)$$

but $u \cdot \nabla \vec{u} = (\nabla \times \vec{u}) \times \vec{u} + \frac{1}{2} \nabla u^2 = \vec{\omega} \times \vec{u} + \frac{1}{2} \nabla u^2$,

where $\vec{\omega} := \nabla \times \vec{u}$ is the vorticity, which describes the local rotational motion of the fluid.

$$\therefore \frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} = -\nabla \left(p + \tfrac{1}{2} u^2 \right) + \nu \nabla^2 \vec{u} + \vec{f}_u \quad (1')$$

If we take the curl of (1'), we find

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega} \times \vec{u}) = \nu \nabla^2 \vec{\omega} + \vec{f}_\omega \quad (2')$$

$$\nabla \times (\vec{\omega} \times \vec{u}) = (u \cdot \nabla) \vec{\omega} - (\omega \cdot \nabla) \vec{u} + \vec{\omega} \underbrace{(\nabla \cdot u)}_{\text{Assumed} = 0} - \vec{u} \underbrace{(\nabla \cdot \omega)}_{\equiv 0}$$

$$\therefore \frac{\partial \vec{\omega}}{\partial t} + \underbrace{(u \cdot \nabla) \vec{\omega}}_{\text{Advection}} = \underbrace{(\omega \cdot \nabla) \vec{u}}_{\text{Vortex stretching}} + \underbrace{\nu \nabla^2 \omega}_{\text{viscous dissipation}} + \underbrace{\vec{f}_\omega}_{\text{External forcing}} \quad (2) \text{ Vorticity Eqn}$$

Equations (1) and (2) are the building blocks for describing hydro turbulence. From these two equations, we will now derive the important conserved quantities in hydro turbulence and see how 3D differs from 2D.

We begin by assuming the system is either periodic and/or homogeneous. The former is true for most simulations, and the latter is true by conjecture and experimental observation.

$\langle f \rangle := \frac{1}{L^3}\int d\vec{x}\, f$ = spatial average

a) $\boxed{\langle f \rangle = 0}$

b) $\boxed{\langle \partial_i^{(n)} f \rangle = 0}$ Fundamental theorem of calc (FTC) and derivatives conserve periodicity

c) $\langle \partial_i (fg) \rangle = 0$ FTC

but $\langle \partial_i (fg) \rangle = \langle f(\partial_i g) \rangle + \langle g(\partial_i f) \rangle \Rightarrow$

$$\boxed{\langle f(\partial_i g) \rangle = -\langle g(\partial_i f) \rangle}$$

d) $\boxed{\langle (\nabla^2 f)\, g \rangle = -\langle (\partial_i f)(\partial_i g) \rangle}$ Integration by parts

e) $\langle \vec{u} \cdot \nabla \times \vec{v} \rangle = \langle \nabla \cdot (u \times v) \rangle + \langle \vec{v} \cdot \nabla \times \vec{u} \rangle$

$$\int d\vec{x}\; \nabla \cdot (u \times v) = \int_S d\vec{S} \cdot (u \times v) = 0$$

$$\therefore \boxed{\langle \vec{u} \cdot \nabla \times \vec{v} \rangle = \langle \vec{v} \cdot \nabla \times \vec{u} \rangle}$$

f) $\langle \vec{u} \cdot \nabla^2 \vec{v} \rangle = ?$

$$\nabla^2 \vec{v} = \nabla(\nabla \cdot \vec{v}) - \nabla \times (\nabla \times \vec{v}) = -\nabla \times (\nabla \times \vec{v}) \quad \text{for } \nabla \cdot \vec{v} = 0$$

$$\vec{u} \cdot \nabla^2 \vec{v} = -\vec{u} \cdot [\nabla \times (\nabla \times \vec{v})] = -(\nabla \times \vec{v}) \cdot (\nabla \times \vec{u}) - \nabla \cdot [\vec{u} \times (\nabla \times \vec{v})]$$

$$\int d\vec{x}\; \nabla \cdot [\vec{u} \times (\nabla \times \vec{v})] = \int_S d\vec{S} \cdot [\vec{u} \times (\nabla \times \vec{v})] = 0$$

$$\therefore \boxed{\langle \vec{u} \cdot \nabla^2 \vec{v} \rangle = -\langle (\nabla \times \vec{u}) \cdot (\nabla \times \vec{v}) \rangle} \quad \text{if } \nabla \cdot \vec{v} = 0$$

Conservation Laws

A) Averaging (1) $\Rightarrow$

$$\partial_t \langle u_i \rangle + \cancelto{0}{\langle \partial_j u_j u_i \rangle} = -\cancelto{0}{\langle \partial_i p \rangle} + \nu \cancelto{0}{\langle \nabla^2 u_i \rangle}$$

by b) by b) by b)

Note that repeated indicies ⇒ summation and

$u_j \partial_j u_i = \partial_j (u_j u_i)$ since $\partial_j u_j = 0$

$\therefore \quad \frac{d\langle \vec{u} \rangle}{dt} = 0$ Conservation of Momentum

B) $\langle \vec{u} \cdot (1) \rangle \Rightarrow$

$$\frac{1}{2}\partial_t \langle u_i^2 \rangle + \cancel{\langle \partial_j (u_j u_i^2/2) \rangle} = -\cancel{\langle \partial_i (u_i p) \rangle} + \nu_i \langle u_i \nabla^2 u_i \rangle$$

by b) by b) f) ⇒ $u_i \nabla^2 u_i = -\langle \vec{\omega}^2 \rangle$

$\therefore \frac{1}{2} \frac{d\langle u^2 \rangle}{dt} = -\nu \langle \omega^2 \rangle$ Conservation of Energy

C) $\langle \vec{u} \cdot (2) \rangle \Rightarrow$

$$\underbrace{\langle \vec{u} \cdot \partial_t \vec{\omega} \rangle}_{i)} + \underbrace{\langle u \cdot [\nabla \times (\omega \times u)] \rangle}_{ii)} = \nu \underbrace{\langle u \cdot \nabla^2 \omega \rangle}_{iii)}$$

i) $\partial_t \langle \vec{u} \cdot \vec{\omega} \rangle = \langle \vec{u} \cdot \partial_t \vec{\omega} \rangle + \langle \vec{\omega} \cdot \partial_t \vec{u} \rangle = 2 \langle \vec{u} \cdot \partial_t \vec{\omega} \rangle$ by e)

ii) $\langle \vec{u} \cdot [\nabla \times (\vec{\omega} \times \vec{u})] \rangle \overset{e)}{=} \langle (\vec{\omega} \times \vec{u}) \cdot (\nabla \times \vec{u}) \rangle = \langle (\vec{\omega} \times \vec{u}) \cdot \vec{\omega} \rangle = 0$

iii) $\langle \vec{u} \cdot \nabla^2 \vec{\omega} \rangle \overset{f)}{=} -\langle (\nabla \times \vec{u}) \cdot (\nabla \times \vec{\omega}) \rangle = -\langle \vec{\omega} \cdot (\nabla \times \vec{\omega}) \rangle$

$\therefore \frac{1}{2} \frac{d\langle \vec{\omega} \cdot \vec{u} \rangle}{dt} = -\nu \langle \omega \cdot (\nabla \times \omega) \rangle$ conservation of Helicity

Summary of general conservation laws

(A) $\frac{d\langle \vec{u} \rangle}{dt} = 0$ Momentum conservation

(B) $\frac{1}{2} \frac{d\langle u^2 \rangle}{dt} =: \frac{dE}{dt} = -\nu \langle \omega^2 \rangle =: -2\nu \Omega$

$E := \frac{1}{2} \langle u^2 \rangle$ = energy ; $\Omega := \frac{1}{2} \langle \omega^2 \rangle$ = enstrophy

$\therefore \frac{dE}{dt} = -2\nu \Omega$ Energy conservation

(C) $\frac{1}{2}\frac{d\langle \vec{u}\cdot\vec{\omega}\rangle}{dt} \Rightarrow \frac{dH}{dt} = -\nu\langle \vec{\omega}\cdot(\nabla\times\omega)\rangle = -2\nu H_\omega$

$H := \frac{1}{2}\langle \vec{u}\cdot\vec{\omega}\rangle$ = helicity; $H_\omega = \frac{1}{2}\langle \vec{\omega}\cdot(\nabla\times\vec{\omega})\rangle$ = vortical helicity

$\therefore \frac{dH}{dt} = -2\nu H_\omega$ Helicity conservation

Finally, what about the possibility of enstrophy conservation?

D) $\langle \vec{\omega}\cdot (2)\rangle \Leftrightarrow$

$$\frac{1}{2}\partial_t\langle \omega_i^2\rangle + \cancel{\langle \partial_j u_j \omega_i^2/2\rangle} = \underbrace{\langle \omega_i \partial_j(\omega_j u_i)\rangle}_{i)} + \nu\underbrace{\langle \omega_i \cdot \nabla^2 \omega_i\rangle}_{ii)}$$

by b)

i) This term is not zero and is a source of enstrophy (vortex stretching).

ii) $\langle \vec{\omega}\cdot\nabla^2\vec{\omega}\rangle = -\langle|\nabla\times\vec{\omega}|^2\rangle$ by f)

$$\therefore \frac{1}{2}\frac{d}{dt}\langle \omega^2\rangle = \langle \vec{\omega}\cdot(\vec{\omega}\cdot\nabla)\vec{u}\rangle - \nu\langle|\nabla\times\vec{\omega}|^2\rangle$$

or $\frac{d\Omega}{dt} = \langle \vec{\omega}\cdot(\vec{\omega}\cdot\nabla)\vec{u}\rangle - 2\nu P$,

where $P := \frac{1}{2}\langle|\nabla\times\vec{\omega}|^2\rangle$ is the palinstrophy

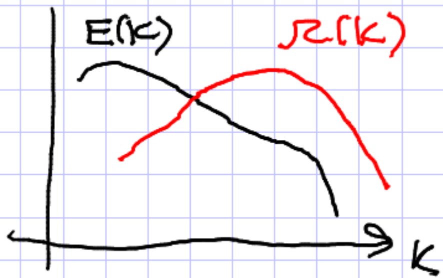
So, the enstrophy is not conserved in 3D. What about 2D?

In 2D (assume xy-plane), $\vec{\omega} = \omega\hat{z} \Rightarrow (\vec{\omega}\cdot\vec{\nabla})\vec{u} \equiv 0$.

Thus, in 2D, enstrophy is conserved, and we have the additional conservation eqn

(D) $\frac{d\Omega}{dt} = -2\nu P$

So, in 2D we have two conserved quadratic quantities, E and Ω, and we might expect there to be cascades of both energy and enstrophy. Since $\vec{\omega} = \nabla \times \vec{u}$, $\omega_k \sim k u_k$. Thus, $\Omega(k) \sim k^2 E(k)$, so Ω should peak at smaller scales than E



In 3D, we assumed that the energy cascade was forward (direct). Let's relax that assumption in 2D and add a friction force to dissipate large scale energy in case there is an inverse cascade.

$$\partial_t \vec{u} + (\vec{u}\cdot\nabla)\vec{u} = -\vec{\nabla} p + \nu \nabla^2 \vec{u} - \underbrace{\alpha \vec{u}}_{\text{friction}} + \vec{f}_u$$

$$\partial_t \vec{\omega} + (\vec{u}\cdot\nabla)\vec{\omega} = \nu \nabla^2 \vec{\omega} - \alpha \vec{\omega} + \vec{f}_\omega$$

As before, define ϵ as the energy transfer rate.

Then, $\underbrace{\epsilon_I}_{\text{Injection}} = \underbrace{\epsilon_\nu}_{\text{viscous diss.}} + \underbrace{\epsilon_\alpha}_{\text{frictional diss.}}$

Similarly, $\eta_I = \eta_\nu + \eta_\alpha$, where η is the enstrophy transfer rate.

$$\frac{\epsilon_\alpha}{\eta_\alpha} \equiv l_\alpha^2 \qquad \frac{\epsilon_\nu}{\eta_\nu} \equiv l_\nu^2 \qquad \frac{\epsilon_I}{\eta_I} = l_f^2$$ are the

characteristic friction, viscous, and driving scales.

These follow from $\Omega(k) \sim k^2 E(k)$.

Note that ϵ_α (η_α) can be viewed as the cascade rate to large scales for the energy (enstrophy) and ϵ_ν (η_ν) as the rate to small scales.
So, lets look at $\frac{\epsilon_\nu}{\epsilon_\alpha}$ and $\frac{\eta_\nu}{\eta_\alpha}$ to determine the directions of the cascade.

$$\frac{\epsilon_f}{\eta_f} = l_f^2 = \frac{\epsilon_\alpha + \epsilon_\nu}{\eta_\alpha + \eta_\nu} = \frac{\frac{\epsilon_\alpha}{\eta_\alpha} + \frac{\epsilon_\nu}{\eta_\alpha}}{1 + \frac{\eta_\nu}{\eta_\alpha}} \Rightarrow$$

$$l_f^2\left(1 + \frac{\eta_\nu}{\eta_\alpha}\right) = \frac{\epsilon_\alpha}{\eta_\alpha} + \frac{\epsilon_\nu}{\eta_\alpha} = l_\alpha^2 + \frac{\epsilon_\nu}{\eta_\nu}\frac{\eta_\nu}{\eta_\alpha} = l_\alpha^2 + l_\nu^2\frac{\eta_\nu}{\eta_\alpha}$$

$$\Rightarrow \boxed{\frac{\eta_\nu}{\eta_\alpha} = \frac{(l_\alpha/l_f)^2 - 1}{1 - (l_\nu/l_f)^2}}$$

$$\frac{\eta_\nu}{\eta_\alpha} = \frac{\eta_\nu/\epsilon_\alpha}{\eta_\alpha/\epsilon_\alpha} = \frac{\eta_\nu/\epsilon_\nu}{\eta_\alpha/\epsilon_\alpha}\frac{\epsilon_\nu}{\epsilon_\alpha} = \frac{l_\alpha^2}{l_\nu^2}\frac{\epsilon_\nu}{\epsilon_\alpha}$$

$$\therefore \boxed{\frac{\epsilon_\nu}{\epsilon_\alpha} = \left(\frac{l_\nu}{l_\alpha}\right)^2 \frac{(l_\alpha/l_f)^2 - 1}{1 - (l_\nu/l_f)^2} = \left(\frac{l_\nu}{l_f}\right)^2\left(\frac{l_f}{l_\alpha}\right)^2 \frac{(l_\alpha/l_f)^2 - 1}{1 - (l_\nu/l_f)^2}}$$

Assuming there is a cascade, it is reasonable to also assume $l_\nu << l_f << l_\alpha$.

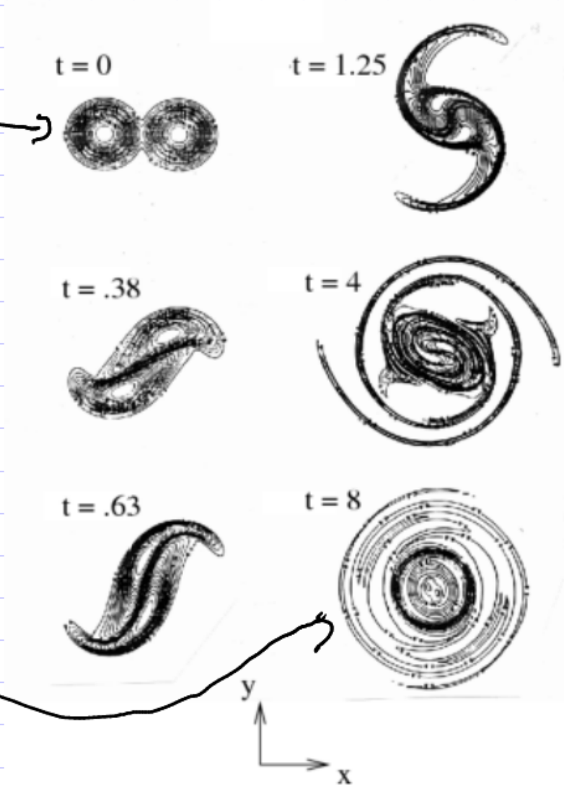
$$\therefore \quad \frac{\epsilon_\nu}{\epsilon_\alpha} << 1 \quad \text{and} \quad \frac{\eta_\nu}{\eta_\alpha} >> 1$$

Thus, more energy cascades to larger than smaller scales (<u>Inverse energy cascade</u>). and more enstrophy cascades to smaller scales than larger (<u>Forward enstrophy cascade</u>).

How can we interpret the inverse cascade?

The simplest, although debated, picture is one of vortex merging

Two initial vortices merge to form one large vortex plus small scale structure



From J.C. McWilliams (1990)

2D Scaling Relations

To derive the scaling relations for 2D turbulence, we'll use essentially the same approach as last time and assumptions from K41

Recall that $E(K) \sim u_K^2/K$ (1D spectrum)

- For scales $l_\alpha > l > l_f$, the energy is cascading inversely $\Rightarrow \epsilon \sim \frac{u^3}{l} \sim K u_K^3 \Rightarrow u_K^2 \sim \epsilon^{2/3} K^{-2/3}$

$\therefore E(K) \sim K^{-5/3} \Rightarrow \Omega(K) \sim K^{1/3}$

The forcing time scale is $t_f \sim \epsilon^{-1/3} l^{2/3}$

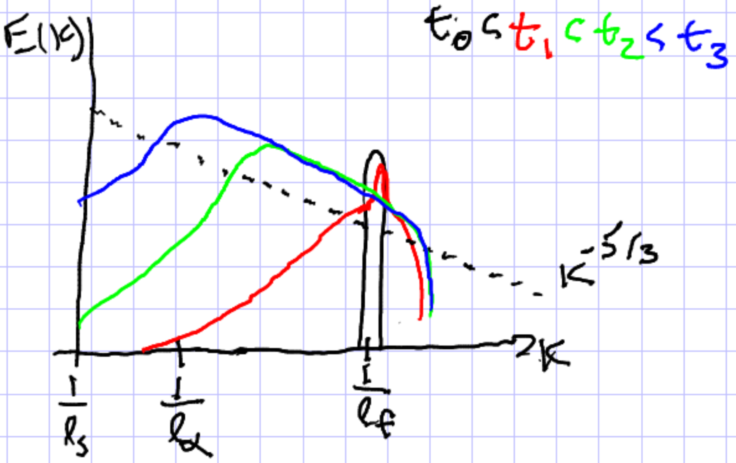
and the time scale associated with friction is

$t_\alpha \sim 1/\alpha$. Equating the times gives the friction cut off scale, $l_\alpha \sim \epsilon^{1/2} \alpha^{-3/2} \sim u_\alpha/\alpha \sim l_f Re_\alpha^{3/2} \gg l_f$,

where $Re_\alpha := \frac{\text{convection}}{\text{friction}} \sim \frac{u_0}{\alpha\, l_f}$ is the friction Reynold's number

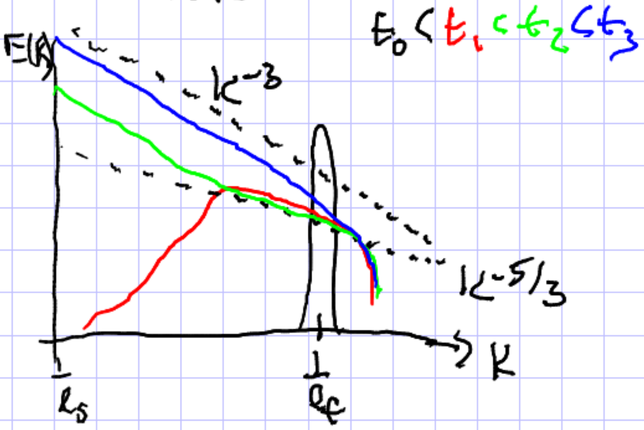
What happens above l_α or if the system size, l_s, exceeds l_α (or $\alpha = 0$)?

Spectral evolution in time with $l_\alpha < l$

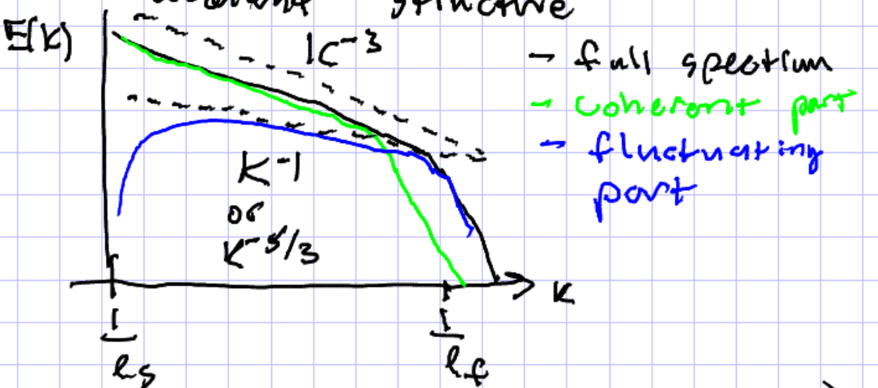


Growth saturates at $l \sim l_\alpha$ due to friction

For $l_\alpha > l_s$, growth ceases at the system scale and energy forms a condensate or system size coherent structure



But, if we subtract the coherent structure

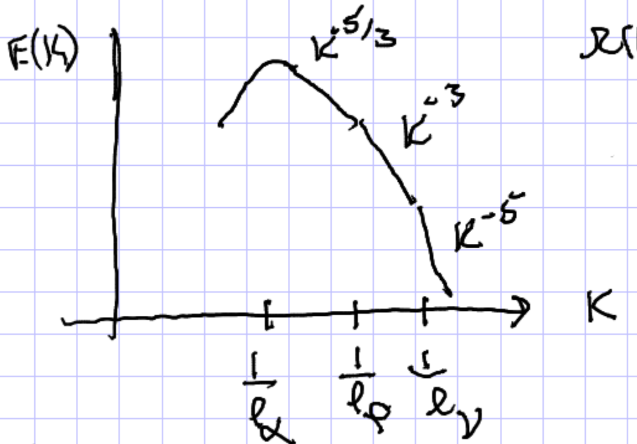


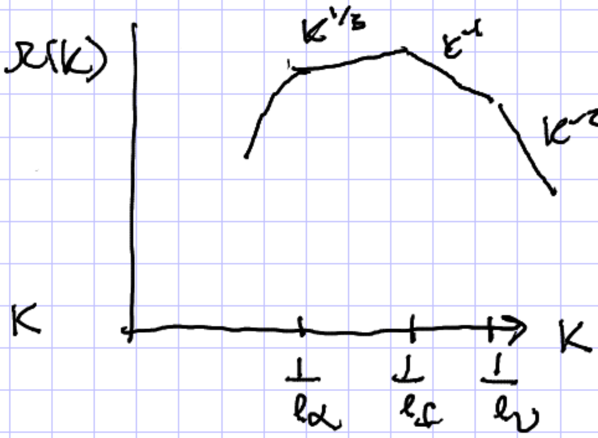
- For $l_f > l > l_\nu$, the enstrophy is actively (forward) cascaded, so we'll focus on the enstrophy cascade rate: $\eta \sim k^2 \epsilon \sim k^3 u^3 \Rightarrow u \sim \eta^{1/3} k^{-1}$

and $E(k) \sim k^{-3} \Rightarrow \Omega(k) \sim k^{-1}$. This steep spectrum implies that shearing from larger scales plays an important role (redo time scale comparison from last lecture to see this). ∴ a non-local correction is necessary. $E(k) \sim k^{-3}[\ln(k/k_{min})]^{-1/3}$

The forcing time scale in terms of η is $t_f \sim \eta^{-1/3}$ and the viscous time scale is, as before, $t_\nu \sim \frac{l^2}{\nu} \Rightarrow l_\nu \sim \eta^{-1/6} \nu^{1/2} \sim l_f Re^{-1/2} \ll l_f$

— Finally, for $l < l_\nu$: $\eta_\nu \sim \nu |\nabla \times \omega|^2 \sim \nu k^4 u^2 \Rightarrow$
$u^2 \sim \eta_\nu / \nu k^{-4}$ $\therefore$ $E(k) \sim k^{-5}$ and $\Omega(k) \sim k^{-3}$





Final remark: Is 2D turbulence even relevant? The answer is yes! When one dimension is much smaller than the other two, that direction is often ignorable. For, instance planetary atmospheres are great examples of 2D turbulence in action. Hurricanes and large scale jet streams are two classic terrestrial examples of 2D turbulence, but so is the great red spot on Jupiter.

Additional reading

1) U. Frisch "Turbulence..." (1996)
Good for more details regarding invariants, but it does not cover much on the way of 2D turbulence. So, see the following for that topic.
2) G. Boffetta and E. Ecke "Two-Dimensional Turbulence" Ann Rev. Fluid Mech (2012) Most readable review
3) R. Kraichnan "Inertial Ranges in Two Dimensional Turbulence" Phys Fluids (1967) First quantitative 2D paper
4) R. Kraichnan and D. Montgomery "Two-dimensional turbulence" Rep. Prog Phys (1980) Quantitative review.

III Diffusion and Passive Scalar Fields

In Lecture I, I stated that turbulence leads to enhanced diffusion. To see this, we will examine the evolution of a passive scalar field, e.g., concentration (density) or temperature, under the influence of a turbulent velocity field.

Let c be the concentration of a substance, eg, dye in water or smoke in air. In the absence of bulk motion, the continuity eqn is simply

$$\frac{\partial c}{\partial t} + \nabla \cdot \vec{j} = 0$$, where $\vec{j}$ is the flux of c

Fick's 1st law of diffusion states that

$\vec{j} = -D_0 \vec{\nabla} c$, ie, the concentration flows down density gradients.

$\therefore \frac{\partial c}{\partial t} = D \nabla^2 c$ This equation is Fick's 2nd law and is a diffusion eqn with

$D \sim V_{th} \lambda_{mfp}$ As an example of molecular (self) diffusion, consider salt deposited at the bottom of a cup (height = 10 cm) of room temperature water.

$D_{NaCl/H_2O} \sim 1.5 \times 10^{-9}\ m^2/s$ and dimensional analysis implies $t \sim \frac{h^2}{D} \sim 77$ days! Self diffusion is very slow.

If we add advection due to the turbulent velocity fluctuations carrying our tracer field, then

$$\frac{\partial c}{\partial t} + \underbrace{\vec{u} \cdot \nabla c}_{\text{advection}} = \underbrace{D \nabla^2 c}_{\text{molecular diffusion}} + \underbrace{S_c}_{\text{source of } c}$$ - Advection-Diffusion (AD) eqn

Now, let's consider concentrations and velocities of the form $c = c_0 + \delta c$ and $\vec{u} = \vec{u}_0 + \delta \vec{u}$ where c_0 and $\vec{u}_0$ are the averages. When measuring the concentration, we are primarily interested in the mean.

Aside: Do we mean a time or space average? Turbulence is typically assumed and seems to obey ergodicity, which means that temporal, spatial, and ensemble averages are equivalent.

Let's average the Advection-Diffusion eqn (AD)

$$\frac{\partial c_0}{\partial t} + \vec{u}_0 \cdot \nabla c_0 = D\nabla^2 c_0 - \nabla \cdot \overset{(*)}{\langle \delta c \, \delta \vec{u} \rangle}$$

Last time, we dropped terms like (*) when averaging to construct conservation equations. This was not actually necessary since (*) is an advection term and the conservation laws were written in terms of convective derivatives. Generally terms like (*) are non-zero. They only vanish if the two variables under the average are uncorrelated.

Let's examine (*) in more detail. (See ① for even more detail)

$\langle \delta c \, \delta \vec{u} \rangle$ is a mass flux just like $\vec{j}$. So, we might reasonably try to rewrite this term as in Fick's 1st law

$\langle \delta c \, \delta u_i \rangle = -D_{t_i} \frac{\partial c_0}{\partial x_i} = -\frac{1}{3} D_t \frac{\partial c_0}{\partial x_i}$ if we assume isotropy.

D_t is the turbulent diffusion coefficient, which is typically called the eddy diffusivity or viscosity. In general, D_t is a $d \times d$ tensor, where $d = \dim \vec{u}$. Also, D_t could be a function of space, but we continue to assume homogeneity.

Thus, the AD eqn becomes

$$\frac{\partial c_0}{\partial t} + \vec{u}_0 \cdot \nabla c_0 = \left(D + \frac{1}{3} D_t\right) \nabla^2 c_0$$

So, turbulence always acts to <u>increase</u> diffusion.

How do D and D_t compare? D_t is dominated by large scale turbulent motions, so $D_t \sim L_c U_{L_c}$. As noted above, $D \sim v_{th} \lambda_{mfp}$. Therefore, $D_t >> D$ because λ_{mfp} is usually very small compared to the system size. From the salt diffusing into water example, $D \sim 1.5 \times 10^{-9} \frac{m^2}{s}$ but $D_t \sim (10 \frac{cm}{s})(10 cm) \sim .01 \frac{m^2}{s}$ with a turbulent diffusion time of $t \sim \frac{h^2}{D_t} \sim 1s$ much faster than molecular diffusion!

All of the above said, it is important to keep in mind that turbulent diffusion does not provide dissipation (increase entropy). It only transfers the variance of the scalar or vector field to small scales. Also, D_t is difficult to evaluate in practice since the concentration gradient depends on the turbulent flow field. So, it is normally evaluated empirically. To derive something a bit more rigorous than the simple estimate $D_t \sim L_c U_{L_c}$ (which is actually a pretty good estimate), consider the displacement of a particle after time t

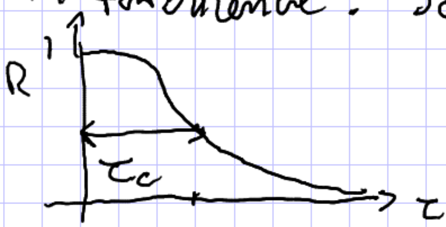
$$\vec{X} = \int_0^t \vec{u} \, d\tau$$

If we consider an ensemble of such particles released from the origin, $\langle \vec{X} \rangle = 0$ but $\langle X^2 \rangle \neq 0$.

(See ② for more)

$\langle X^2(t) \rangle = 2 \int_0^t \int_0^{t'} \langle u(t) u(t+\tau) \rangle d\tau dt'$, where $\frac{\langle u_i(t) u_j(t+\tau) \rangle}{\langle u^2 \rangle}$ is the autocorrelation tensor, R_{ij}, introduced in Lecture I. To keep things simple, we will consider just 1 dimension, eg, isotropic diffusion.

One would expect R to tend to zero for long times since the particle has a short "memory" in turbulence. So, R should resemble



For short times, the velocity is strongly correlated. We can define a correlation time in the same way that I defined the correlation or integral length scale in lecture I

$\tau_c := \int_0^\infty R \, d\tau$. The correlation length is simply $L_c := \sqrt{\langle u^2 \rangle}\, \tau_c$, where $\sqrt{\langle u^2 \rangle}$ is the RMS.

So, for $t < \tau_c$ and $l < L_c$ velocities are strongly correlated and $R \sim 1$.

- For $t \gg \tau_c$, $\langle x^2 \rangle = 2 \langle u^2 \rangle \tau_c t + \text{const.}$

$\Rightarrow$ the RMS particle separation is

$\sqrt{\langle x^2 \rangle} \sim t^{1/2}$. $\therefore$ the separation is analogous to a <u>random walk</u> because the steps are uncorrelated.

The rate of change of the variance of X is proportional to the diffusion coefficient (see ③)

$$D_t := \frac{1}{2} \frac{d \langle x^2 \rangle}{dt} = \langle u^2 \rangle \tau_c = \sqrt{\langle u^2 \rangle}\, L_c,$$ which

is similar to my initial estimate from dimensional analysis.

Note that the above applies for large systems only. Specifically, $L \gg L_c$, where L is the integral length

- For $t \ll \tau_c$, $R \sim 1 \Rightarrow \langle x^2 \rangle = \langle u^2 \rangle t^2$

and $\sqrt{\langle x^2 \rangle} \sim t$, i.e., correlated stepping

$$D_t \approx \frac{1}{2} \frac{d \langle x^2 \rangle}{dt} = \langle u^2 \rangle t$$ not constant in time

for short travel times.

Note that in previous lectures, I defined ℓ as the size of an eddy, but this could just as well be interpreted as the RMS particle separation of a tracer field. So, $\ell \sim \sqrt{\langle x^2 \rangle}$ and $\frac{d\ell}{dt} \sim \delta u_\ell$, where we can interpret δu_ℓ as the velocity difference between particles.

Now, we can derive scalings for small scale turbulent diffusion.

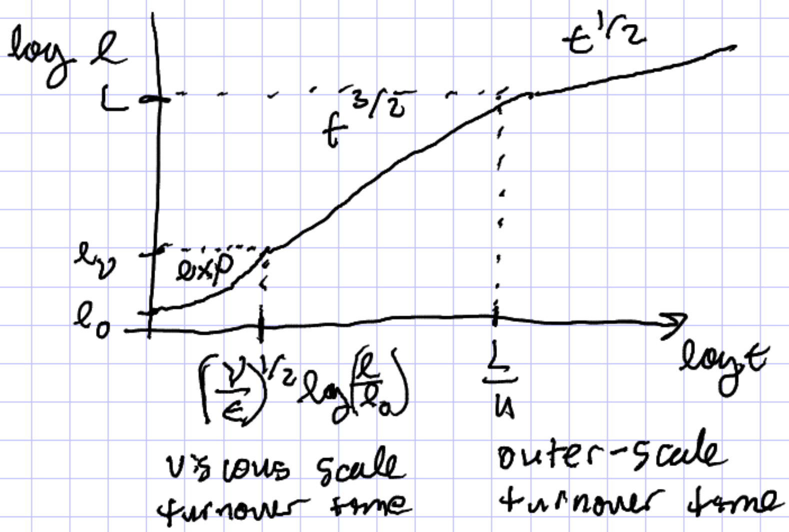
- In the inertial range, $\ell_\nu < \ell < L$, $\delta u_\ell \sim (\epsilon \ell)^{1/3} \sim \frac{d\ell}{dt}$

$$\Rightarrow \ell \sim \sqrt{\epsilon}\, t^{3/2} \quad \text{- Richardson law}$$

- For $\ell < \ell_\nu$, $\delta u_\ell \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} \ell \sim \frac{d\ell}{dt}$

$$\Rightarrow \ell \sim \ell_0\, e^{\sqrt{\epsilon/\nu}\, t} \Leftrightarrow \text{exponential separation}$$

So, we have



Although I noted earlier that $D \ll D_t$, molecular diffusion must become important at some scale so as to provide dissipation. Let's try to find that scale.

First, let's derive the diffusion version of the Reynold's number, which is the Peclet number,

$$P_e := \frac{\text{Advection}}{\text{Diffusion}} = \frac{u \cdot \nabla C}{D \nabla^2 C} \sim \frac{\delta u_{L_c} L_c}{D}$$, where L_c is the length scale of the source of c. If we consider the same ratio at smaller scales, then when "P_e" ~ 1, molecular diffusion becomes dominant.

If this occurs in the inertial range, $\delta u_\ell \sim (\epsilon \ell)^{1/3}$

$\therefore \; \ell_D \sim \left(\frac{D^3}{\epsilon}\right)^{1/4} \sim P_e^{-3/4} L_c$ where ℓ_D is the molecular diffusion scale.

Note that $P_e \gg 1$ typically, so $\ell_D \ll L_c$

We can also write ℓ_D in terms of the viscous scale, ℓ_ν, from lecture I

$$\ell_D \sim \left(\frac{D^3}{\epsilon}\right)^{1/4} \sim \left(\frac{D}{\nu}\right)^{3/4} \ell_\nu ,$$

$Sc := \frac{\nu}{D}$ is the Schmidt number.

$\therefore \; \ell_D \sim Sc^{-3/4} \ell_\nu$

Since we assumed inertial range scaling for δu_ℓ, this is only valid for $\ell_D \gg \ell_\nu$, ie, $Sc \ll 1$

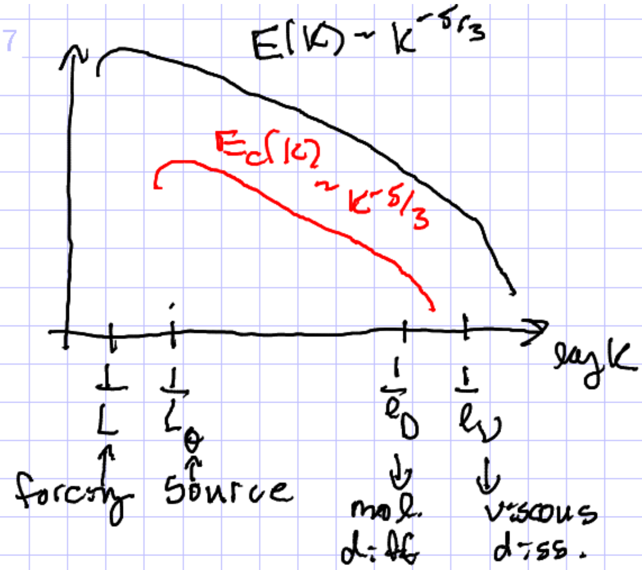
If we make the same assumptions as in K41, then $\epsilon_c \sim \frac{\delta c_\ell^2}{t_\ell} \sim$ const. flux of scalar variance

But the turbulence obeys $\epsilon \sim \frac{\delta u_\ell^2}{t_\ell}$

$$\Rightarrow \delta c_\ell^2 \sim \frac{\epsilon_c}{\epsilon} \delta u_\ell^2 \sim \frac{\epsilon_c}{\epsilon^{1/3}} \ell^{2/3}$$

$$\therefore \; E_c(k) \sim \frac{\epsilon_c}{\epsilon^{1/3}} k^{-5/3}$$ Obukhov-Corrsin spectrum

traces the turbulence



This is only the case for $Sc = \frac{\nu}{D} \ll 1$, ie, $\nu \ll D$. Sc for air mixtures is usually $0.2 < Sc < 3$. So, this would be the spectrum for a tracer in air

What about the case $Sc \gg 1$, ie, $\nu \gg D$?

For scales $\ell > \ell_\nu$, the scaling is as in the previous case.

For $\ell < \ell_\nu$, we assume the viscous scaling for $\delta u_\ell \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} \ell$

Then, $\frac{\delta u_\ell \ell}{D} \sim \frac{\left(\frac{\epsilon}{\nu}\right)^{1/2} \ell^2}{D} \sim 1 \Rightarrow \ell_D \sim \left(\frac{D^2 \nu}{\epsilon}\right)^{1/4} \sim Sc^{-1/2} \ell_\nu$

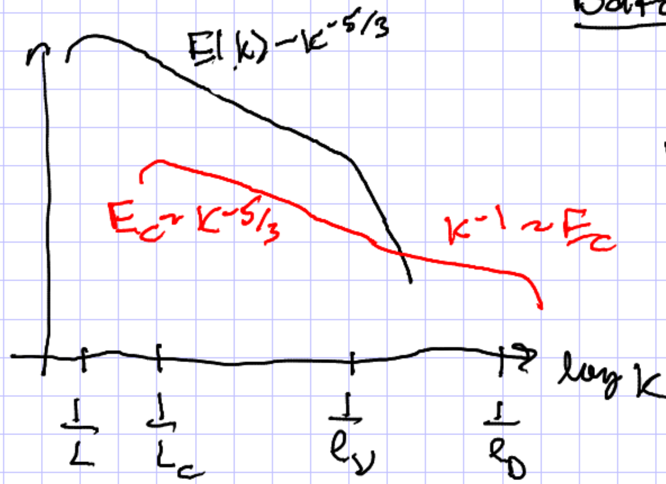
$\Rightarrow \ell_D \ll \ell_\nu$

For this new range of scales, $\ell_\nu \gg \ell \gg \ell_D$, the turbulence obeys the viscous range scalings

So, $\epsilon_c \sim \frac{\delta c_\ell^2}{t_\ell}$ and $t_\ell \sim \frac{\ell}{\delta u_\ell} \sim \left(\frac{\nu}{\epsilon}\right)^{1/2}$

$\Rightarrow \delta c_\ell^2 \sim \epsilon_c \left(\frac{\nu}{\epsilon}\right)^{1/2} \Rightarrow E_c(K) \sim \epsilon_c \left(\frac{\nu}{\epsilon}\right)^{1/2} K^{-1}$

<u>Batchelor spectrum</u>



Valid for $Sc \gg 1$, ie, $\nu \gg D$. Sc for water mixtures are typically a few hundred. So, this would be the spectrum for tracers in water

① A semi-rigorous derivation of turbulent diffusion

We need an expression for $\nabla \cdot \langle \delta c\, \delta \vec{u} \rangle$

As in the body of the lecture, let $c = \underbrace{c_0}_{\text{mean}} + \underbrace{\delta c}_{\text{fluctuating}}$ and $\vec{u} = \vec{u}_0 + \delta \vec{u}$. Neglecting molecular diffusion, the AD eqn is $\frac{\partial c}{\partial t} + u \cdot \nabla c = 0$

$$\frac{\partial c_0}{\partial t} + \frac{\partial \delta c}{\partial t} + u_0 \cdot \nabla c_0 + \delta u \cdot \nabla c_0 + u_0 \cdot \nabla \delta c + \delta u \cdot \nabla \delta c = 0 \quad (1)$$

Averaging (1) $\Rightarrow$ $\frac{\partial c_0}{\partial t} + u_0 \cdot \nabla c_0 = -\nabla \cdot \langle \delta \vec{u}\, \delta c \rangle \quad (2)$

(1)−(2) $\Rightarrow$ $\frac{\partial \delta c}{\partial t} + u_0 \cdot \nabla \delta c + \delta u \cdot \nabla \delta c = \frac{d \delta c}{dt} = -\delta u \cdot \nabla c_0 + \nabla \cdot \langle \delta \vec{u}\, \delta c \rangle$

Integrating the total derivative,

$$\delta c = -\int_0^t d\tau\, \delta u(\tau) \cdot \nabla c_0(\tau) + \int_0^t d\tau\, \nabla \cdot \langle \delta \vec{u}(\tau)\, \delta c(\tau) \rangle$$

$$\Rightarrow \langle \delta c(t)\, \delta \vec{u}(t) \rangle = -\langle \delta \vec{u}(t) \int_0^t d\tau\, \delta \vec{u}(\tau) \cdot \nabla c_0(\tau) \rangle$$

$$+ \underbrace{\langle \overbrace{\delta \vec{u}(t)} \int_0^t d\tau\, \nabla \cdot \langle \delta \vec{u}(\tau)\, \delta c(\tau) \rangle \rangle}_{\langle \left(\text{fluctuating quantity}\right) * \left(\text{Averaged quantity}\right) \rangle = 0}$$

$$\Rightarrow \langle \delta c\, \delta \vec{u} \rangle = -\int_0^t d\tau\, \langle \delta \vec{u}(t)\, \delta \vec{u}(\tau) \rangle \cdot \nabla c_0(\tau)$$

Several things happened here. $\delta \vec{u}(t)$ commuted with the integral, the average passed into the integral, and the average was split between $\langle \delta \vec{u}\, \delta \vec{u} \cdot \nabla c_0 \rangle = \langle \delta u\, \delta u \rangle \cdot \nabla c_0$. The first two are straightforward and the third follows because c_0 is already averaged and is not correlated with $\delta \vec{u}$.

But $\langle \delta u_i(t)\, \delta u_j(\tau) \rangle = \langle u^2 \rangle R_{ij}(t-\tau) = S_{ij}(t-\tau)$, where R_{ij} is the autocorrelation tensor previously defined.

Define $t' = t - \tau$, then

$$\langle \delta c\, \delta u_i \rangle = -\int_0^t dt'\, S_{ij}(t')\, \partial_j c_0(t-t')$$

Taylor expanding, $c_0(t-t') = c_0(t) - \underbrace{t' \frac{\partial}{\partial t} c_0(t) + \ldots}_{\text{small for } t \gg \tau_c}$

$\therefore C_0(t-t') \sim C_0(t)$ and

$$\langle \delta c \delta u_i \rangle = -\left[\int_0^t dt' \, S_{ij}(t')\right] \partial_j C_0(t)$$

For $t \gg \tau_c$, $\int_0^t dt' \, S_{ij}(t') =: D_{ijt}$ = const (space independent too because of homogeneity)

$\therefore \langle \delta c \delta u_i \rangle = -\frac{1}{3} D_{ijt} \, \partial_j C_0 \rightarrow$ Fick's 1st law

Note that D_{ijt} is manifestly positive definite.

D_{ijt} is the turbulent diffusion coefficient. Isotropy $\Rightarrow D_{ijt} = \frac{1}{3}\delta_{ij} D_t$

So, $\nabla \cdot \langle \delta c \delta \vec{u} \rangle = -\frac{1}{3} D_t \nabla^2 C_0$. □

See T.C. Lipscombe et al "On the Convection of a Passive Scalar by a Turbulent Gaussian Velocity Field" (1991) for a more formal proof.

(2) The derivation of $\langle x^2 \rangle = 2\int_0^t\int_0^{t'} \langle u(t) u(t+\tau) \rangle d\tau \, dt'$ is sketched below.

$x(t') = \int_0^{t'} u \, d\tau \qquad u = \frac{dx}{dt'}$

$$\therefore \left\langle x \frac{dx}{dt'} \right\rangle = \frac{1}{2}\frac{d\langle x^2 \rangle}{dt'} = \left\langle \left(\int_0^{t'} u \, d\tau\right) u \right\rangle$$

$$= \int_0^{t'} \langle u(\tau) u(t) \rangle d\tau$$

$$\Rightarrow \langle x^2 \rangle = 2\int_0^t \int_0^{t'} \langle u(\tau+t) u(t) \rangle d\tau \, dt'$$

This is not a rigorous proof. See G.I. Taylor "Diffusion by continuous Movements" (1921) for the (gory) details.

(3) From (2), $\frac{1}{2}\frac{d\langle x^2 \rangle}{dt'} = \int_0^{t'} \langle u(t) u(\tau) \rangle d\tau$,

but the latter expression is precisely how I defined the turbulent diffusion coefficient.

$$\therefore \frac{1}{2}\frac{d\langle x^2 \rangle}{dt} = D_t$$

IV Correlation Functions and the $\frac{4}{5}$ Law

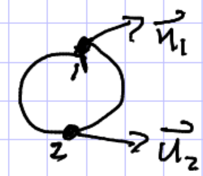
Up to this point, we've used the assumptions of K41 to develop a variety of scaling relations to describe aspects of turbulence but no attempt has been made to "solve" turbulence. Today, we will make an attempt to solve the NS eqns, $\nabla \cdot \vec{u} = 0$ and $\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \nu \nabla^2 \vec{u} + \vec{f}_u$. But $\vec{u}$ is a random field, $\vec{f}_u$ is usually assumed to be random (necessary for the following description), and there may be statistically random initial conditions. So, solving for a particular realization of $\vec{u}$ is not useful or typically possible. What we usually want in "solving" turbulence are universal solutions. So, our goal should be to solve for average quantities or statistical distributions.

Before we attempt to solve turbulence, I list the symmetries of the NS eqn (absent forcing & boundaries)

- Space-translations: $t, \vec{r}, \vec{u} \mapsto t, \vec{r}+\vec{r}', \vec{u}$
- Time-translations: $t, \vec{r}, \vec{u} \mapsto t+t', \vec{r}, \vec{u}$
- Parity: $t, \vec{r}, \vec{u} \mapsto t, -\vec{r}, -\vec{u}$
- Rotations: $t, \vec{r}, \vec{u} \mapsto t, R\vec{r}, R\vec{u}$ $\quad R \in SO(\mathbb{R}^3)$
- Scaling: $t, \vec{r}, \vec{u} \mapsto \lambda^{1-h} t, \lambda \vec{r}, \lambda \vec{u}$ $\quad \lambda > 0, \lambda \in \mathbb{R}, h \in \mathbb{R}$
- Galilean invariance (*): $t, \vec{r}, \vec{u} \mapsto t, \vec{r} + \vec{u}_0 t, \vec{u} + \vec{u}_0$

(*) The NS eqn obeys Galilean invariance; however, $\vec{u}_0$ breaks isotropy by adding a preferred direction. If we assume $\vec{u}_0$ is random and isotropically distributed, isotropy is retained. Importantly, third-order structure functions remain invariant (why this is important will become clear at the end of the lecture). This form of invariance is called random Galilean invariance.

Given the above symmetries, I will use $\vec{u}$ to be a fluctuating quantity, i.e., $\langle \vec{u} \rangle = 0$. Although inconsistent with notation from lecture III, but consistent with previous lectures, $\delta \vec{u} = \vec{u}(\vec{x}_2) - \vec{u}(\vec{x}_1)$
$= \vec{u}_2 - \vec{u}_1$



Using the velocity, we can construct two second order average quantities (moments)

① $\langle u_{1i} u_{2j} \rangle =: C_{ij}(\vec{x}_1, \vec{x}_2) \underset{\substack{\uparrow \\ \text{Homogeneity}}}{=} C_{ij}(\vec{x}_2 - \vec{x}_1) = C_{ij}(\vec{r})$

Recall that $R_{ij} = \frac{\langle u_{1i} u_{2j} \rangle}{\langle u^2 \rangle}$ was defined as the normalized autocorrelation tensor. $C_{ij} = \langle u_{1i} u_{2j} \rangle$ is simply the <u>2nd order 2 point correlation tensor.</u>

② $\langle \delta u_i \, \delta u_j \rangle = S_{ij}(\vec{r})$ is the <u>2nd order 2point structure function</u>

$C_{ii}(0)$ and $S_{ii}(0)$ are 2nd order 1pt moments.

Note that $C_{ii}(0) = \langle u^2 \rangle = 2E$ and $C_{ij}(0) = \underset{\substack{\uparrow \\ \text{isotropy}}}{(\text{const})} \delta_{ij}$

$\Rightarrow \text{const} = \frac{2}{3} E$ or $C_{ij}(0) = \frac{2}{3} E \delta_{ij}$

In general, we can construct m-order n-point statistical correlation functions, eg, $\langle u_{1i} u_{2j} u_{3k} \rangle = C_{ijk}$, where $0 < n \leq m$.

The general form for an isotropic rank 2 tensor, C_{ij}, is

$C_{ij}(\vec{r}) = \alpha(r)\delta_{ij} + \beta(r)\hat{r}_i \hat{r}_j + \gamma(r)\cancel{\hat{r}_k \epsilon_{ijk}}^{\,0}$, where $\hat{r}_i = r_i / r$

Parity $\Rightarrow C_{ij}(-\vec{r}) = C_{ij}(\vec{r})$ (also follows from permutation symmetry)

$\therefore C_{ij}$ can be written as

$$C_{ij}(\vec{r}) = \underset{\substack{\uparrow \\ \text{transverse}}}{C_{TT}} (\delta_{ij} - \hat{r}_i \hat{r}_j) + \underset{\substack{\uparrow \\ \text{longitudinal}}}{C_{LL}} \hat{r}_i \hat{r}_j$$

Note that $S_{ij}(\vec{r}) = S_{TT}(\delta_{ij} - \hat{r}_i \hat{r}_j) + S_{LL} \hat{r}_i \hat{r}_j = \langle \delta u_i \, \delta u_j \rangle =$

$$\underset{\substack{\shortparallel \\ C_{ij}(0)}}{\langle u_{2i} u_{2j} \rangle} + \underset{\substack{\shortparallel \\ C_{ij}(0)}}{\langle u_{1i} u_{1j} \rangle} - \underset{\substack{\shortparallel \\ C_{ij}(\vec{r})}}{\langle u_{1i} u_{2j} \rangle} - \underset{\substack{\shortparallel \\ C_{ji}(\vec{r})}}{\langle u_{2i} u_{1j} \rangle}$$

$$= 2C_{ij}(0) - 2C_{ij}(\vec{r}) = \frac{2}{3} \langle u^2 \rangle \delta_{ij} - 2C_{ij}(\vec{r})$$

$$\therefore S_{TT} = \frac{2}{3} \langle u^2 \rangle - 2C_{TT} \quad ; \quad S_{LL} = \frac{2}{3} \langle u^2 \rangle - 2C_{LL}$$

Note that $C_{ii}(0) = \langle u^2 \rangle = 2C_{TT}(0) + C_{LL}(0)$ but $S_{ii}(0) = 0$.

Up to this point, we haven't used the NS eqns, so let's see if they can provide any useful info.

Consider $\nabla \cdot \vec{u} = 0$, ie $\partial u_i / \partial x_i = 0$

Well $\dfrac{\partial C_{ij}}{\partial x_{2j}} = \left\langle u_{1i} \dfrac{\partial u_{2j}}{\partial x_{2j}} \right\rangle = 0$. We can also

write $\dfrac{\partial}{\partial x_{2j}} = \dfrac{\partial r_j}{\partial x_{2j}} \dfrac{\partial}{\partial r_j} = \dfrac{\partial}{\partial r_j} = \hat{r}_j \dfrac{\partial}{\partial r}$. Similarly, $\dfrac{\partial}{\partial x_{1j}} = -\dfrac{\partial}{\partial r_j} = -\hat{r} \dfrac{\partial}{\partial r_j}$

$$\text{So, } \frac{\partial C_{ij}}{\partial x_{2j}} = \frac{\partial}{\partial x_{2j}} \left[C_{TT}(\delta_{ij} - \hat{r}_i \hat{r}_j) + C_{LL} \hat{r}_i \hat{r}_j \right]$$

$$= C_{TT}' \underbrace{\hat{r}_j (\delta_{ij} - \hat{r}_i \hat{r}_j)}_{=0} + C_{LL}' \underbrace{\hat{r}_i \hat{r}_j \hat{r}_j}_{\hat{r}_i} + (C_{LL} - C_{TT}) \frac{\partial}{\partial r_j} (\hat{r}_i \hat{r}_j)$$

$$\frac{\partial}{\partial r_j} (\hat{r}_i \hat{r}_j) = \frac{1}{r^2} (r_j \delta_{ij} + r_i \delta_{jj}) - \frac{2}{r^3} \hat{r}_i \hat{r}_j \hat{r}_j = \frac{\hat{r}_i}{r} 2$$

$$\therefore \hat{r}_i \left[C_{LL}' + \frac{2}{r} (C_{LL} - C_{TT}) \right] = 0 \quad \text{or}$$

$$C_{LL}' + \frac{2C_{LL}}{r} = \boxed{\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 C_{LL}) = \frac{2}{r} C_{TT}}$$

von Karman relation

This means we only need one scalar function.

An analogous relation holds for S_{LL} and S_{TT}

Incompressibility gave us the von Karman relation. What can the NS eqn do for us?

$$\partial_t \langle u_{1i} u_{2j} \rangle = \langle u_{1i} \partial_t u_{2j} \rangle + \langle u_{2j} \partial_t u_{1i} \rangle =$$

$$\langle -u_{1i} u_{2k} \frac{\partial u_{2j}}{\partial x_{2k}} - u_{1i} \frac{\partial p_2}{\partial x_{2j}} + \nu u_{1i} \frac{\partial^2 u_{2j}}{\partial x_2^2} + u_{1i} f_{2j} \rangle +$$

$$\langle - u_{2j} u_{1k} \frac{\partial u_{1i}}{\partial x_{1k}} - u_{2j} \frac{\partial p_1}{\partial x_{1i}} + \nu u_{2j} \frac{\partial^2 u_{1i}}{\partial x_1^2} + u_{2j} f_{1i} \rangle =$$

$$-\frac{\partial}{\partial x_{1k}} \langle u_{1i} u_{1k} u_{2j} \rangle - \frac{\partial}{\partial x_{2k}} \langle u_{1i} u_{2k} u_{2j} \rangle \quad (3^{rd} \text{ order correlations!})$$

$$-\frac{\partial}{\partial x_{1i}} \langle p_1 u_{2j} \rangle - \frac{\partial}{\partial x_{2j}} \langle p_2 u_{1i} \rangle$$

$$+\nu \nabla_1^2 \langle u_{1i} u_{2j} \rangle + \nu \nabla_2^2 \langle u_{1i} u_{2j} \rangle$$

$$+\langle u_{1i} f_{2j} \rangle + \langle u_{2j} f_{1i} \rangle$$

So, we have $\partial_t \langle uu \rangle \sim \nabla \cdot \langle uuu \rangle$, and we would find that $\partial_t \langle uuu \rangle \sim \nabla \cdot \langle uuuu \rangle \Rightarrow$ a closure problem analogous to collisionless plasma fluid closures. This closure problem is generic to averaged nonlinear equations.

Let's ignore the closure issue for the moment and continue.

Well, 1) $\langle u_{1i} u_{2j} \rangle = C_{ij}(\vec{r})$

2) $\langle u_{1i} u_{1k} u_{2j} \rangle = C_{ik,j}(\vec{r})$ – 3^{rd} order 2 point correlation function $\Rightarrow$ symmetry in the ik indices.

Parity also implies $C_{ik,j}(-\vec{r}) = -C_{ik,j}(\vec{r})$ because it is an odd function (same for all odd orders). Also, we can write

$\langle u_{1i} u_{2k} u_{2j} \rangle = C_{kj,i}(\vec{r})$. Finally,

$\langle u_{1i} u_{1k} u_{2j} \rangle = -\langle u_{2i} u_{2k} u_{1j} \rangle$ by interchanging points 1 and 2 and taking $\vec{r} \to -\vec{r}$.

3) $\langle p_2 u_{1i} \rangle = 0$ because

$$\frac{\partial}{\partial x_{2j}} \langle p_1 u_{2j} \rangle = \langle p_1 \frac{\partial u_{2j}}{\partial x_{2j}} \rangle = 0 \text{ by incompressibility}$$

By homogeneity and isotropy, we can assume $\langle p_1 u_{2j} \rangle = f(r)\hat{r}_j$

$$\Rightarrow \frac{\partial}{\partial x_{2j}} \langle p_1 u_{2j} \rangle = \frac{\partial}{\partial r_j}\left(f(r)\hat{r}_j\right) = f' + \frac{f}{r}\delta_{ii} - f\hat{r}_j \frac{r_j}{r^2} = f' + \frac{2f}{r} = 0$$

$\Rightarrow f = \frac{const}{r^2}$, but $f(r \to 0)$ is finite $\Rightarrow$ const $= 0$

$\therefore \langle p_1 u_{2j} \rangle = \langle p_2 u_{1i} \rangle = 0$

4) $\langle u_{1i} f_{2j} \rangle = \epsilon_{ij}$ the 2 point version of the injected energy flux.

Putting 1)-4) together, we have

$$\partial_t C_{ij} = \frac{\partial}{\partial r_k}\left(C_{ik,j} + C_{jk,i}\right) + 2\nu \nabla^2 C_{ij} + 2\epsilon_{ij} \quad (1)$$

We showed earlier that C_{ij} is a function of just one scalar (von Karman relation). So, we can just look at the trace of (1) without losing any information

So, $$\frac{1}{2}\partial_t C_{ii} = \frac{\partial}{\partial r_k} C_{ik,i} + \nu \nabla^2 C_{ii} + \epsilon_{ii} \quad (2)$$

(2) is the 2 point version of the energy equation.

$$C_{ii} = 2C_{TT} + C_{LL} = 2C_{LL} + rC_{LL}' + C_{LL} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 C_{LL}\right),$$

but $C_{LL} = \frac{1}{3}\langle u^2 \rangle - \frac{1}{2}S_{LL} \Rightarrow$

$$C_{ii} = \langle u^2 \rangle - \frac{1}{2r^2}\frac{\partial}{\partial r}\left(r^3 S_{LL}\right)$$

Still need an expression for $C_{ik,i}$. The general form for a rank 3 tensor symmetric in its first two indices is

$$C_{ik,j} = \alpha(r)\delta_{ik}\hat{r}_j + \beta(r)\left(\delta_{ij}\hat{r}_k + \delta_{kj}\hat{r}_i\right) + \gamma(r)\hat{r}_i\hat{r}_j\hat{r}_k$$

(Same as heat flux in gyrotropic plasma) playing the same trick using incompressibility

$\Rightarrow C_{LLT} = C_{TTT} = 0$, $C_{LTT} = \frac{1}{6}\frac{\partial}{\partial r}(r C_{LLL})$ and

the same for $S_{LTT}, S_{LLL}, S_{LLT}, S_{TTT}$. The end result

being $C_{ik,j}(\vec{r}) = -\frac{1}{12}\left[S_{LLL}\,\delta_{ik}\hat{r}_j - \frac{1}{2}(r S'_{LLL} + 2S_{LLL})(\delta_{ij}\hat{r}_k + \delta_{kj}\hat{r}_i) \right.$
$\left. + (r S'_{LLL} - S_{LLL})\hat{r}_i\hat{r}_j\hat{r}_k \right]$

(See Landau & Lifshitz Fluid Mechanics §34 for details)

So, $C_{ik,i} = -\frac{\hat{r}_k}{12}\left[S_{LLL} - \frac{1}{2}(r S'_{LLL} + 2S_{LLL})(3+1) + r S'_{LLL} - S_{LLL} \right]$

$= \frac{1}{12}\hat{r}_k (r S'_{LLL} + 4S_{LLL}) = \frac{1}{12}\frac{\hat{r}_k}{r^3}\frac{\partial}{\partial r}(r^4 S_{LLL})$

Putting C_{ii} and $C_{ik,i}$ into ② gives the

<u>von Karman - Howarth equation</u> relating S_{LL} to S_{LLL},

but this is just a restatement of the closure

problem. If we assume stationarity, $\partial_t C_{ii} = 0$,

and large scale energy injection $\Rightarrow$

$\epsilon_{ii}(r) = \epsilon_{ii}(0) + \ldots$ Taylor expansion for $r = 0$

and $\epsilon_{ii}(0) \approx \epsilon$ (conventional energy flux).

then ② becomes $\frac{\partial}{\partial r_k}\left[C_{ik,i} + \nu \frac{\partial}{\partial r_k} C_{ii} \right] = -\epsilon_{ii}$

Integrating $\Rightarrow$ $C_{ik,i} + \nu \frac{\partial}{\partial r_k} C_{ii} = -\frac{1}{3}\epsilon r_k + \text{const}$

$\frac{1}{3}$ from $\frac{\partial r_k}{\partial r_k} = \delta_{kk}$ and const $= 0$ because @ $r_k = 0$

$\frac{\partial}{\partial r_k} C_{ii}(0) = 0$ and $C_{ik,i}(0) = 0$ because odd

power of u.

So, we have $C_{ik,i} + \nu \frac{\partial C_{ii}}{\partial r_k} =$

$\frac{1}{12}\frac{\hat{r}_k}{r^3}\frac{\partial}{\partial r}(r^4 S_{LLL}) - \nu\, \hat{r}_k \frac{\partial}{\partial r}\left(\frac{1}{2r^2}\frac{\partial}{\partial r} r^3 S_{LL}\right) =$

$$\frac{1}{12}\frac{\hat{r}_k}{r^3}\frac{\partial}{\partial r}\left[r^4\left(S_{LLL}-6\nu S_{LL}'\right)\right] = -\frac{1}{3}\epsilon r_k$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left[r^4\left(S_{LLL}-6\nu S_{LL}'\right)\right] = -4\epsilon r^4$$

Integrating $\Rightarrow$ $$\boxed{S_{LLL} = -\frac{4}{5}\epsilon r + 6\nu S_{LL}'}$$

This is Kolmogorov's 4/5 law and the only exact, non-trivial result. Often, $6\nu S_{LL}'$ is dropped by assuming $r \gg l_\nu$ (or $Re \to \infty$)

Several caveats must now be stated

1) Isotropy and homogeneity were assumed. These can be relaxed, but the 4/5 becomes a function of direction.

2) I explicitly assumed stationarity, $r \gg l_\nu$, and $r \ll L_0$. Formally, these must be done in the correct order

$$\lim_{r\to 0}\lim_{\nu\to 0}\lim_{t\to\infty}\frac{S_{LLL}}{r} = -\frac{4}{5}\epsilon.$$ If $r \to 0$ before $\nu \to 0$, then we would have found $S_{LLL} \sim \text{Const}\,\epsilon r^3$ because $\delta u \sim r$ for $l < l_\nu$. If $\nu \to 0$ before $t \to \infty$, singularities appear in the NS eqn.

3) The 4/5 law does not provide proof that the cascade is local, or that a Kolmogorov spectrum emerges.

4) The 4/5 law is not invariant to Galilean transformations, but it is invariant to random Galilean transformations discussed at the beginning of the lecture. (See Frisch for more details about random Galilean transforms)

5) the above derivation required a force, but the force breaks the Galilean (and random) invariance of the NS eqn. that's ok, because the correlation appearing and used in the result, $\langle u f \rangle$, do preserves it.

6) I assumed a forcing term, but what about decaying turbulence for which $\epsilon_{ii} \approx 0$? Well, we still assume that $\partial_t S_{LL} = 0$ (this is self-similar decay), but $\partial_t \langle u^2 \rangle \neq 0$ because the energy is decaying in time. So, the "source" term becomes $\partial_t \langle u^2 \rangle = -\epsilon$. With this in mind, ϵ in the 4/5 law can simply be viewed as the energy flux at an inertial range scale, i.e., ϵ = const.

7) A result similar to the 4/5 law exists for passive scalars

$$\hat{r} \cdot \langle \delta c^2 \delta \vec{u} \rangle = -\frac{4}{3} \epsilon_c r + 2D \frac{\partial}{\partial r} \langle c^2 \rangle$$ – Yaglom's law