V Closure Models, an Overview

Last lecture we saw that in attempting to "solve" turbulence we uncovered a closure problem whereon $\partial_t \langle u^n \rangle \sim \nabla \cdot \langle u u^n \rangle$. This time we will examine the problem in more detail and various closure models that have been developed. All of these models are ad hoc and should be taken with a (large) grain of salt. They are all wrong, but given enough adjustable parameters, they can be made to work well.

As usual, lets start with the NS eqn

$$\frac{\partial \vec{U}}{\partial t} + \vec{U} \cdot \nabla \vec{U} = -\nabla P + \nu \nabla^2 \vec{U} \quad (1), \text{ where } \vec{U} = \underset{\text{mean}}{\vec{u}_0} + \underset{\text{fluctuating}}{\vec{u}}$$

$P = p_0 + p$, $\nu \nabla^2 \vec{U} = \nabla \cdot (2\nu \underline{\underline{S}})$, $\underline{\underline{S}} = \frac{1}{2}\left[(\nabla \vec{U}) + (\nabla \vec{U})^T\right]$,

$\underline{\underline{S}}$ is the rate of strain tensor, and $\underline{\underline{S}} = \underline{\underline{s}}_0 + \underline{\underline{s}}$.

Averaging (1)
$$\frac{\partial \vec{u}_0}{\partial t} + \vec{u}_0 \cdot \nabla \vec{u}_0 = -\vec{\nabla} p_0 + \nabla \cdot (2\nu \underline{\underline{s}}_0 + \langle \vec{u}\vec{u} \rangle)$$

$\langle \vec{u}\vec{u} \rangle$ is the 2^{nd} order 1 point correlation function from the last lecture, $C_{ij}(0)$. This term, $\langle u_i(\vec{x}) u_j(\vec{x}) \rangle$ is often called the Reynold's stress, R_{ij}.

So, we have
$$\frac{\partial \vec{u}_0}{\partial t} + \vec{u}_0 \cdot \nabla \vec{u}_0 = -\vec{\nabla} p_0 + \nabla \cdot (2\nu \underline{\underline{s}}_0 + \underline{\underline{R}}). \quad (2)$$

The mean flow depends on the turbulence through a second order moments. So, we need an equation for $\underline{\underline{R}}$

Subtracting ①-② yields

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}_0\cdot\nabla\vec{u} + \vec{u}\cdot\nabla\vec{u}_0 + \nabla\cdot(\vec{u}\vec{u} - \underline{\underline{R}}) = -\nabla p + \nu\nabla^2\vec{u}$$

So, $\partial_t(u_i u_j) = u_j \partial_t u_i + u_i \partial_t u_j = -u_j u_{0k}\partial_k u_i - u_i u_{0k}\partial_k u_j$

$-u_j u_k \partial_k u_{0i} - u_i u_k \partial_k u_{0j} - u_j \partial_k(u_k u_i) - u_i \partial_k(u_k u_j) + u_j \partial_k R_{ik}$
$+ u_i \partial_k R_{jk} - u_j \partial_i p - u_i \partial_j p + \nu(u_i \nabla^2 u_j + u_j \nabla^2 u_i)$

Averaging this and simplifying leads to

$$\partial_t \underline{\underline{R}} + \vec{u}_0\cdot\nabla\underline{\underline{R}} + \underline{\underline{R}}\cdot\nabla\vec{u}_0 + (\underline{\underline{R}}\cdot\nabla\vec{u}_0)^T + \nabla\cdot(\vec{u}\vec{u}\vec{u}) = 2\langle p\underline{\underline{s}}\rangle + \nabla\langle p u\rangle$$
$$+ (\nabla\langle p u\rangle)^T + \nu\nabla^2\underline{\underline{R}} + 2\nu\langle(\partial_k\vec{u})(\partial_k\vec{u})\rangle \quad ③$$

and we see that the equation for $\underline{\underline{R}}$ depends on third order moments, $\nabla\cdot(\vec{u}\vec{u}\vec{u})$. Note that equation ①-③ display the closure problem using 1 pt statistics. Last lecture we developed the equations for 2 pt statistics, which we will re-visit shortly.

Let us now explore a simple 1 pt closure. Note that 1 pt closures are sometimes referred to as "engineering" closures since they typically prescribe a model for $\underline{\underline{R}}$ and thus only provide information about the mean flow, which is often sufficient for engineering purposes.

The basis for most of these closures was introduced by Boussinesq in 1887

$-\underline{\underline{R}} = 2\nu_T \underline{\underline{S}}_0 - \frac{2}{3}\underline{\underline{I}}E$, where $E = \frac{1}{2}\langle u^2\rangle$ is the familiar kinetic energy, $\underline{\underline{S}}_0$ is the mean rate of strain tensor and ν_T is a prescribed coefficient. ν_T is typically interpreted as an eddy viscosity because inserting this closure into eqn ② ⇒

$$\frac{\partial \vec{u}_0}{\partial t} + \vec{u}_0 \cdot \nabla \vec{u}_0 = -\vec{\nabla}\left(p_0 + \tfrac{2}{3}E\right) + \nabla \cdot \left[2(\nu+\nu_T)\underline{\underline{S}}_0\right]$$

A simple estimate for $\nu_T = \langle u^2 \rangle^{1/2} L_m$, where L_m is the mixing length, which can be experimentally determined. L_m is analogous to the mean free path but for a fluid parcel. This simple estimate works passably well sometimes, but obviously fails if for instance the turbulence is anisotropic. More advanced models such as Spalart-Allmaras, $k-\epsilon$, and $k-\omega$ models employ different and more complex estimates for ν_T, but they all only work well for specific cases, and none of these actually solve for turbulent quantities.

So, let's now move on to the two point version of ③, which we derived last time (dropping the driving term)

$$\frac{1}{2}\partial_t C_{ii} = \frac{\partial}{\partial r_k} C_{ik,i} + \nu \nabla^2 C_{ii} \quad ④$$

Schematically, this is simply

$$\partial_t \langle uu \rangle = \langle uuu \rangle + \nu \langle uu \rangle \quad ⑤$$

the third order equation is

$\partial_t \langle uuu \rangle = \langle uuuu \rangle + \langle uup \rangle + \nu \langle uuu \rangle$, but p can be computed by taking the divergence of the NS eqn, $p(\vec{x}') = \frac{1}{4\pi}\int \frac{1}{|\vec{x}-\vec{x}'|}\frac{\partial^2 u_m u_n}{\partial x_m \partial x_n} d\vec{x}$ ⇒

$$\partial_t \langle uuu \rangle = \langle uuuu \rangle + \int \langle uuuu \rangle + \nu \langle uuu \rangle \quad ⑥$$

the first 2pt closure we'll explore (and revisit later) is the quasi-normal (QN) closure. This style of closure assumes that the 2pt joint PDF of u is Gaussian — Gaussian PDFs of $u \Rightarrow$ that $u_i(\vec{x})$ and $u_j(\vec{x}')$ are statistically independent. This is approximately true for well separated ($r \gg L$) points but usually (*) not true at smaller scales. Nonetheless, Millionshtchikov proposed such a closure in 1941, where (5) and (6) are kept in their exact form and $\langle u_i u_j' u_k'' u_l''' \rangle$ is approximated as $\langle u_i u_j' \rangle \langle u_k'' u_l''' \rangle + \langle u_i u_k'' \rangle \langle u_j' u_l''' \rangle + \langle u_i u_l''' \rangle \langle u_k'' u_j' \rangle$ due to the QN assumption. A different way to state this is that the 4th order cumulants are zero. This approach closes the system, but it has some serious deficiencies. First, the system is time reversible, which is inconsistent with turbulence. Second, as the system evolves, the energy spectrum develops large, negative portions, which is unphysical. Refinements to QN can be incorporated, and we will discuss them shortly. To do so, we need to move over to Fourier space.

Recall that $C_{ij}(r)$ and $C_{ik,j}$ are each functions of single scalar functions, eg,

$\frac{1}{2} C_{ii}(r) = C(r)$ and $C_{ik,i} = D(r)\,\hat{r}_k$

$\therefore$ (4) can be rewritten as simply

$$\partial_t C(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 D(r)\right] + 2\nu\nabla^2 C(r) \quad (7)$$

This is one way to write the von Karman-Howarth equation mentioned but not written last time.

Before going further with the closure discussion, we need to convert (7) to Fourier space.

 (*) Wind tunnel experiments do have quasi-Gaussian PDFs

$$u_i(\vec{k}) = \int d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}}\, u_i(\vec{x})$$

$$u_i(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, u_i(\vec{k})$$

$$\Rightarrow \langle u_i(\vec{k}) u_j(\vec{k}')\rangle = \int d\vec{x} \int d\vec{x}'\, e^{-i\vec{k}\cdot\vec{x}-i\vec{k}'\cdot\vec{x}'}\, \underbrace{\langle u_i(\vec{x})\, u_j(\vec{x}')\rangle}_{C_{ij}(\vec{r})}$$

$$= \underbrace{\int d\vec{r}\, e^{-i\vec{k}\cdot\vec{r}}\, C_{ij}(\vec{r})}_{C_{ij}(\vec{k})} \underbrace{\int d\vec{x}'\, e^{-i\vec{x}'\cdot(\vec{k}+\vec{k}')}}_{(2\pi)^3\delta(\vec{k}+\vec{k}')}$$

$$\therefore \langle u_i(\vec{k})\, u_j(\vec{k}')\rangle = C_{ij}(\vec{k})\,(2\pi)^3\, \underbrace{\delta(\vec{k}+\vec{k}')}_{\text{because of homogeneity}}$$

Isotropy & parity $\Rightarrow$

$$C_{ij}(\vec{k}) = A(k)\,\hat{k}_i\hat{k}_j + B(k)\,\delta_{ij}\,, \quad \hat{k}_i = \frac{k_i}{k}$$

Incompressibility $\Rightarrow$ $k_i C_{ij} = 0 \Rightarrow B = -A$

or $\quad C_{ij}(\vec{k}) = B(k)(\delta_{ij} - \hat{k}_i\hat{k}_j)$

Note that $\frac{1}{2} C_{ii} = B(k)$ and

$$E = \frac{1}{2}\langle u^2\rangle = \frac{1}{2}\int C_{ii}(\vec{k})\, e^{i\vec{k}\cdot\vec{0}}\, d\vec{k} = \int B(k)\, d\vec{k} = \int_0^\infty 4\pi k^2 B(k)\, dk$$

but this means that $B(k) \equiv E^{(3)}(k)$, the 3D energy spectrum or that $4\pi k^2 B(k) = E(k)$, the 1D energy spectrum.

$$\therefore C_{ij}(\vec{k}) = \frac{E(k)}{4\pi k^2}(\delta_{ij} - \hat{k}_i\hat{k}_j)$$

Now, Fourier transforming (7) with the above derivation and some algebra the spectral version of the von Karman-Howarth equation emerges

$$\partial_t E(k) = T(k) - 2\nu k^2 E(k),$$

where T is basically the Fourier transform of $\frac{1}{r^2}\frac{\partial}{\partial r}[r^2 D(r)]$ and is called the spectral kinetic energy transfer function. Alternatively, we could write $\partial_t E = -\dfrac{\partial \Pi}{\partial k} - 2\nu k^2 E$, where

$\Pi = -\int_0^K T(k)dk = \int_K^\infty T(k)dk$ is the spectral kinetic energy flux and represents the net transfer of energy from all eddies with $k' < k$ to wavenumbers $k'' > k$ — $\Pi = \epsilon$ in the inertial range

So, now that we are in Fourier space, rather than looking for a closure that relates $\langle uuu \rangle$ to $\langle uu \rangle$, we want a relationship between $\Pi(k)$ and $E(k)$ or a dynamical equation of the form $\partial_t T(k) = \cdots$. The former are called algebraic closures and the QN closure is an example of the latter. Let's first look at some algebraic closures.

Algebraic closures are constructed to guarantee that the correct inertial range spectrum is recovered. So, in that sense, they all work well, but all of the early closures, eg, Obukhov (1941), Ellison (1961), and Heisenberg (1948), all suffered from the obvious deficiency that Π depended on a large range of k. For instance, Heisenberg (1948) proposed

$$\Pi = \alpha \int_k^\infty k^{-3/2} E^{1/2} dk \int_0^k k^2 E \, dk$$

This obviously violates locality in wavenumber.

The most successful of these closures is due to Pao (1965) $\quad \Pi = \alpha \epsilon^{1/3} k^{5/3} E$

This closure actually works quite well. It is local and produces an energy spectrum

$E = \alpha \epsilon^{2/3} k^{-5/3} \exp[-\frac{3\alpha}{2}(k/k_\nu)^{4/3}]$, which works in both the inertial range and dissipation range.

the Pao model even works relatively well at predicting the spectrum for scales $K < 1/L$. But, the Pao model fails to accurately predict things like the skewness and flatness of the velocity PDFs. So, let's now revisit and improve on the QN closure introduced earlier. As before, we'll retain the exact 2nd order equation

$$\frac{\partial E}{\partial t} = T - 2\nu K^2 E$$ and the 3rd order eqn

which is of the form $\frac{\partial T}{\partial t} = NL \text{ terms} + \text{viscous terms}$.

So, let's now derive the 3rd order eqn, but first we need some notation.

The 3rd order, 3 point velocity correlation tensor

$$C_{i,j,k}(\vec{r},\vec{r}\,') = \langle u_i(\vec{x})\, u_j(\vec{x}\,')\, u_k(\vec{x}\,'') \rangle = \langle u_i\, u_j'\, u_k'' \rangle$$

$\vec{r} = \vec{x}\,' - \vec{x}$ and $\vec{r}\,' = \vec{x}\,'' - \vec{x}$. The Fourier xform of $C_{i,j,k}(\vec{r},\vec{r}\,')$ is

$$\Phi_{ijk}(\vec{K},\vec{K}') = \frac{j}{(2\pi)^6} \iint C_{i,j,k}\, e^{-j(\vec{K}\cdot\vec{r} + \vec{K}'\cdot\vec{r}\,')}\, d\vec{r}\, d\vec{r}\,'$$

and we can write T in terms of Φ_{ijk} as

$T = 4\pi K^2 k_j \int \Phi_{iij}(\vec{K},\vec{K}')\, d\vec{K}'$. The NS eqn $\Rightarrow$

$$\left[\partial_t + \nu(K^2 + K'^2 + K''^2)\right]\Phi_{ijk} = \text{Transform}[\langle uuuu \rangle],$$ where

$\vec{K} + \vec{K}' + \vec{K}'' = 0$ (*). Using the QN hypothesis (and a lot of algebra) one finds

$$\begin{aligned}\left[\partial_t + \nu(K^2 + K'^2 + K''^2)\right]\Phi_{ijk} = {} & P_{i\alpha l}(\vec{K}'')\,\Phi_{\alpha j}(\vec{K})\,\Phi_{\alpha l}(\vec{K}') + \\ & P_{j\alpha l}(\vec{K})\,\Phi_{\alpha k}(\vec{K}')\,\Phi_{li}(\vec{K}'') + \\ & P_{k\alpha l}(\vec{K}')\,\Phi_{\alpha i}(\vec{K}'')\,\Phi_{\alpha j}(\vec{K})\end{aligned}$$

(*) This is a general property of turbulence we will revisit where $P_{i\alpha}(\vec{k}) = k_\ell \Delta_{i\alpha}(\vec{k}) + k_\alpha \Delta_{i\ell}(\vec{k})$ and

$\Delta_{i\alpha}(\vec{k}) = \delta_{i\alpha} - k_i k_\alpha / k^2$

To simplify things, we'll just write

$$[\partial_t + \nu(k^2 + k'^2 + k''^2)]\Phi_{ijk} = \sum_{QN} \Phi_{ij}\Phi_{ij} \qquad (8)$$

When (8) was first derived, the community was excited, but it suffers the same problems as the previous QN closure we discussed. Importantly, it leads to significant negative values in the energy spectrum. That QN fails isn't very surprising since the joint PDFs are typically far ~~from~~ from Gaussian. Part of the reason for the failure is that the QN theory builds in an arrow of time [Orszag (1970)]. Obviously, a viscous fluid is irreversible, but Orszag argued that even inviscid fluids should be statistically irreversible since on average they lead to greater mixing as time proceeds. So, he proposed adding an eddy viscosity to the 3^{rd} order equation, creating the eddy damped QN (EDQN) closure,

$$[\partial_t + \nu(k^2 + k'^2 + k''^2) + \mu(\vec{k}) + \mu(\vec{k}') + \mu(\vec{k}'')]\Phi_{ijk} = \sum_{QN} \Phi_{ij}\Phi_{ij}$$

where $\mu(\vec{k}) \sim [k^3 E(k)]^{1/2}$ (same as we derived for turbulent diffusion in lecture III)

or sometimes $\mu(\vec{k}) \sim [\int_0^k k'^2 E(k')dk']^{1/2}$.

With this modification, the EDQN model is irreversible even when $\nu \equiv 0$.

Another point EDQN addresses is the following: In QN, for ν small and k not too large ($k < k_\nu$)

$\Phi_{ijk} = \int_0^t \sum_{QN} \Phi_{ij} \Phi_{ij} d\tau$, which implies that the instantaneous triple correlation depends on the entire history of the double correlation product. this is not very plausible. If we rewrite the EDQN closure as

$$\underbrace{[\partial_t + 1/\theta]}_{\text{all viscous terms}} \Phi_{ijk} = \sum_{QN} \Phi_{ij} \Phi_{ij} \quad \text{, then}$$

$\Phi_{ijk} = \int_0^t [\exp[(\tau - t)/\theta] \sum_{QN} \Phi_{ij} \Phi_{ij}] d\tau$. This makes it clear that in the EDQN closure Φ_{ijk} only depends on the immediate history of $\Phi_{ij} \Phi_{ij}$ and that θ is equivalent to the relaxation time. Note that θ is always of order the eddy turnover time.

An alternate way to view the μ terms are as a model for the 4th order cumulant terms discarded in QN theory, ie, the joint PDFs are no longer purely Gaussian. But the EDQN model still fails to guarantee that the energy spectrum remains positive! Orszag "fixed" this in the late seventies by Markovianizing the EDQN model -- this just means to remove the memory of the system so that the conditional PDFs of future states depend only on the present state. So, the EDQNM drops the time derivative of Φ_{ijk} leaving

$$[\nu(k^2 + k'^2 + k''^2) + \mu(\vec{k}) + \mu(\vec{k}') + \mu(\vec{k}'')] \Phi_{ijk} = \sum_{QN} \Phi_{ij} \Phi_{ij}$$

this model finally gets the spectrum correct and is somewhat justifiable for small, fast eddies compared to larger, slower eddies. $\therefore$ EDQNM should not be applied to the energy containing range. EDQNM is still very popular and remains one of the best closure models. Two other popular, advanced models are the direct-interaction-approximation (DIA) [Kraichnan (1959)] (it doesn't even get the correct $k^{-5/3}$, though) and the test field model (TFM). TFM is very similar to EDQNM, but it uses an auxiliary (test) field to evaluate θ.

In summary, there are many ways to close the 1pt and 2pt equations, and they all work to varying degrees, but they all also suffer serious deficiencies.

Further reading

1) P.A. Davidson "Turbulence" This book covers all of the closures discussed (and more) in more detail.

2) M. Lesieur "Turbulence in Fluids" covers the EDQNM closure in great detail. You can also get a free digital copy from the UMD library website

VI Intermittency

Up to this point, we have adopted the clean picture of turbulence presented in K41. In reality self-similarity, homogeneity, and constant cascade rates do not hold. All of this is related to the concept of intermittency. Before we discuss intermittency, we need to establish some basic tools.

A) Probability density functions

Since we are now going to be discussing turbulence in a more purely statistical sense, we will be using the probability density (or distribution) function, f, of a random variable, in our case the velocity, u. The probability $u \in (a,b)$ is given by

$$P(a<u<b) = \int_a^b f(u)\,du \quad \text{and} \quad \int_{-\infty}^{\infty} f\,du = 1$$

Effectively, f contains all of the information about the turbulence. Using f, we can construct some very important quantities:

Mean $\mu = \int_{-\infty}^{\infty} u f\,du = \langle u \rangle$

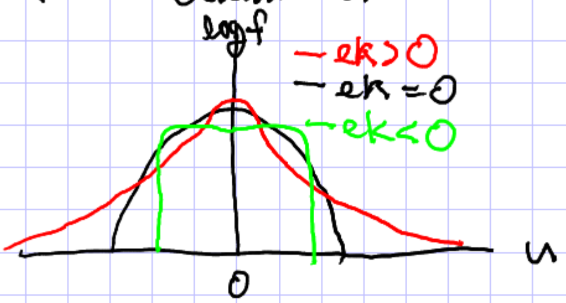
Variance $\sigma^2 = \int_{-\infty}^{\infty} (u-\mu)^2 f\,du$

or if we consider only distributions with $\mu = 0$, which is a standard convention and what we use moving forward

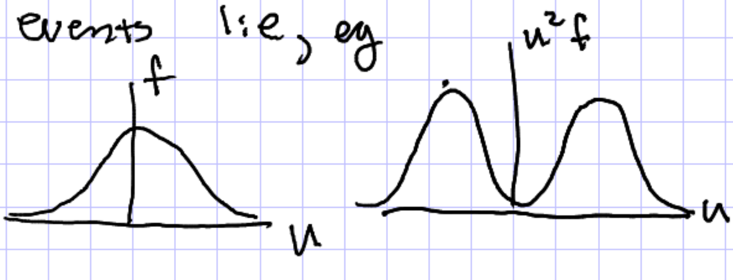
$\sigma^2 = \int_{-\infty}^{\infty} u^2 f\,du = \langle u^2 \rangle$ this measures the spread of the data

Skewness $S = \dfrac{\int_{-\infty}^{\infty} u^3 f\,du}{\sigma^3} = \dfrac{\langle u^3 \rangle}{\langle u^2 \rangle^{3/2}}$ this measures the asymmetry of f. It is the first moment that actually tells us something about the shape of f. Note that a unimodal distribution can still be asymmetric but have $S=0$ if for instance the left tail is long and thin and the right tail is short and fat such that they have the same area.

Flatness or Kurtosis $K = \frac{\int_{-\infty}^{\infty} u^4 f du}{\sigma^4} = \frac{\langle u^4 \rangle}{\langle u^2 \rangle^2}$ This measures the flatness or peakedness of f — this also effectively gives us information about the fatness of the tails. A normal distribution has $K=3$, so a quantity called excess kurtosis is often defined as $ek = K-3$ for comparison with normal distributions. $ek < 0$ are called platykurtic and $ek > 0$ are leptokurtic (heavier tales than Gaussian).



Arbitrarily high order moments can be constructed but may not be finite (In fact, even those described above can diverge). The higher the moment of f, the more weight is given to the tails of f, where infrequent events lie, eg



Completely analogous quantities to those above can be defined for velocity increments, $\delta u(r) = u(x+r) - u(x)$.

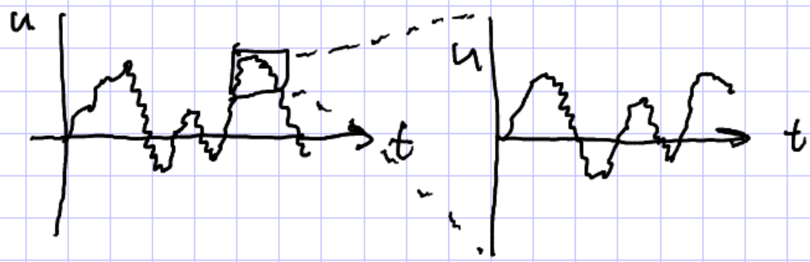
 Further, in the limit $r \to 0$, the same quantities exist for derivatives, eg,

$$S(r\to 0) = S_0 = \frac{\langle \delta u^3 \rangle}{\langle \delta u^2 \rangle^{3/2}} = \frac{\langle (\partial u/\partial x)^3 \rangle}{\langle (\partial u/\partial x)^2 \rangle^{3/2}}$$

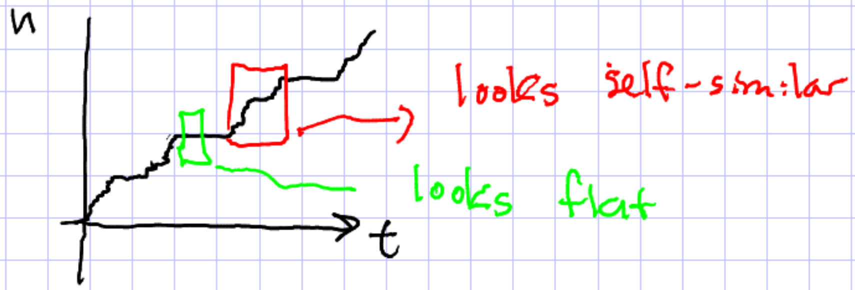
B Intermittency

Now that we have the tools, lets examine intermittency

We all know intuitively what intermittency means. If a process is self-similar, eg, Brownian motion, it is not intermittent

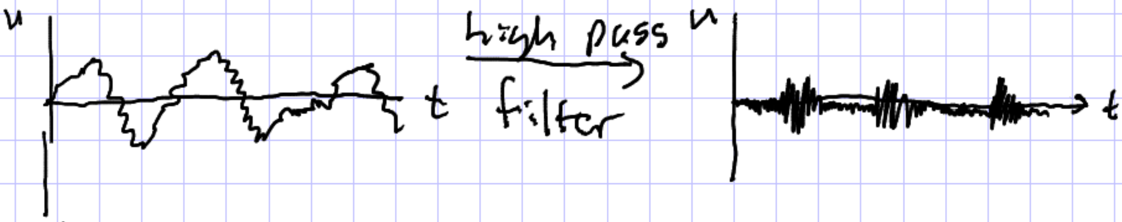


A classic intermittent example is the devil's staircase



The smaller the fragment we examine, the more likely it will be flat.

An intermittent signal could be "hiding" in data that looks self-similar



This is how intermittency in turbulence was identified, and related to one way to define intermittency

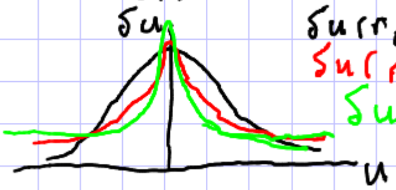
$$u(t) = \int_{\mathbb{R}} d\omega\, e^{i\omega t}\, u(\omega) \leftarrow \text{full FT}$$

$$u^{>}_{\Omega}(t) = \int_{\omega>\Omega} d\omega\, e^{i\omega t}\, u(\omega) \leftarrow \text{high passed FT}$$

Then a signal is intermittent if

$$K(\Omega) = \frac{\langle u^{>}_{\Omega}(t)^4 \rangle}{\langle u^{>}_{\Omega}(t)^2 \rangle^2} \quad \text{grows without bound with } \Omega$$

In practice, it's more common to use structure functions since they already have a scale associated with them. Note that $f(u)$ is actually very close to Gaussian because the value of u at any particular point is largely determined by chance. However, either filtered u or more commonly $\delta u(r)$ is highly non-Gaussian for large Ω or small r. This is often visualized as

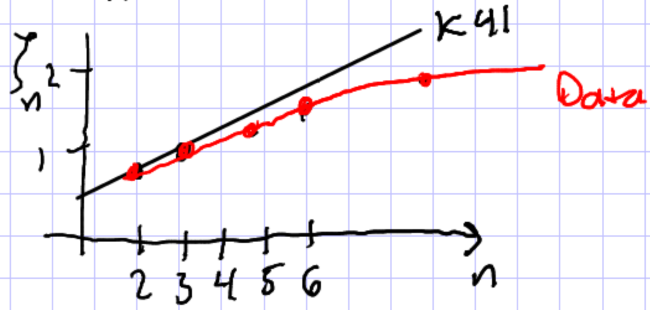


or

The non-Gaussian nature of δu follows because it is related to the local dynamics.

As noted above, the higher order the moment considered, the more focused is the quantity on the tails, which is where the extreme/intermittent events live. So, we can also look for intermittency by looking at scaling exponents of the structure functions:

$$S_n(r) = \langle |\delta\vec{u}\cdot\hat{r}|^n \rangle \sim r^{\zeta_n} \quad \text{for } l_\nu \ll r \ll L$$



Note that: ζ_n vs n must be both concave and non-decreasing (Frisch §8.4).

The $\frac{4}{5}$ law tells us that $\langle \delta u^3 \rangle = S_3(r) = -\frac{4}{5}\epsilon r$ $\Rightarrow$ $\zeta_3 = 1$. Note that the $4/5$ law also implies that the skewness of turbulence <u>must</u> be negative, which immediately $\Rightarrow$ that f is <u>not</u> Gaussian.

i) The K41 prediction

K41 assumes self-similarity $\Rightarrow$ $\delta u(\lambda r) = \lambda^h \delta u(r)$

$\Rightarrow$ $S_n(\lambda r) = \lambda^{hn} S_n(r)$ $\forall n$. Well, we know

$S_3 = -\frac{4}{5}\epsilon r$ $\therefore$ $h = 1/3$ and $\zeta_n = n/3$. Thus,

we get the straight line on the ζ_n vs n figure for K41. This is also referred to as being monofractal, eg, 1 vortex always splits into two (self-similarity), which is equivalent to ϵ_r = constant.

ii) Refined Similarity [Kolmogorov 1962 (K62)]

We could consider a modification of K41 based on intermittency, which is related to the volume or time filling of the field. So, lets consider the possibility that S_n originates from just some fraction of the volume, $\chi_r^{(n)} \sim \left(\frac{r}{L}\right)^{\mu_n}$ Then

$$S_n(r) = \langle \delta u_r^n \rangle \sim \delta u_r^n \chi_r^{(n)} \sim (\epsilon r)^{n/3}\left(\frac{r}{L}\right)^{\mu_n} \sim \epsilon^{n/3} L^{-\mu_n} r^{\frac{n}{3}+\mu_n}$$

$\Rightarrow$ $\zeta_n = \frac{n}{3} + \mu_n$ (μ_n must be a function of n since otherwise, $\zeta_3 = 1$ $\Rightarrow$ $\mu = 0$).

Kolmogorov in 1962 explained the origin of the lack of volume filling as follows

Consider an energy flux at a given scale

$$\epsilon_r(\vec{x}) = \frac{1}{\frac{4}{3}\pi r^3}\int_{|\vec{x}'-\vec{x}|<r} d\vec{x}'\, \nu |\nabla u(\vec{x}')|^2$$

By this definition,

$\langle \epsilon_r \rangle = \epsilon$ = traditional (K41) energy flux

Such local energy flux would be the case of dissipation primarily occurs on filamentary structures.

If there is a local energy flux, then
$\delta u_r \sim (\epsilon_r r)^{1/3}$ rather than $\delta u_r \sim (\epsilon r)^{1/3}$ in K41.
$\langle \epsilon_r^n \rangle = \epsilon^n \left(\frac{r}{L}\right)^{\mu_n}$, where $\mu_1 = 0$
$S_n(r) = \langle \delta u_r^n \rangle \sim \langle (\epsilon_r r)^{n/3} \rangle = \langle \epsilon_r^{n/3} \rangle r^{n/3} = \epsilon^{n/3} L^{-\mu_n/3} r^{\frac{n}{3}+\mu_n}$
$\therefore \zeta_n = \frac{n}{3} + \mu_n$, where μ_n is the scaling exponent for ϵ_r. This is the refined similarity hypothesis.

Since ϵ_r is no longer constant, we are now considering multifractal* turbulence and global scale invariance has become local, i.e., $\delta u_{\lambda r} = \lambda^{h(r)} \delta u_r$ (loosely, h actually depends on the fractal dimension)

*Turbulence can be seen to be multifractal from the ζ_n vs n plot. Each time the fractal dimension changes, the slope of the figure changes. Since it is a roughly smooth curve, it is referred to as multifractal (continuous spectrum of fractal dimensions).

c) Modelling Intermittency

Now that we know turbulence is intermittent, let's develop a way to model and describe the intermittency. To do so, we'll use the simplest and first model developed

The Random Cascade Model

Consider a system of size L. Then $\langle \epsilon_L \rangle = \epsilon$. Now, divide the box into smaller boxes of size αL, $\alpha < 1$. Then, because of intermittency, we can say that $\epsilon_{\alpha L} = \epsilon W_1$, where W_1 is a random variable, $W_1 \geq 0$ and $\langle W_1 \rangle = 1$ because $\langle \epsilon_{\alpha L} \rangle = \epsilon$. Now, divide the boxes again $\epsilon_{\alpha^2 L} = \epsilon_{\alpha L} W_2 = \epsilon W_1 W_2'$, where W_2' is independent of W_1, but distributed the same but $\langle \epsilon_{\alpha^2 L} \rangle = \epsilon$. Continuing to divide to a scale $l = \alpha^k L$, $\epsilon_l = \epsilon W_1 W_2 \cdots W_k$, $\langle \epsilon_l \rangle = \epsilon$. Then $\langle \epsilon_l^m \rangle = \epsilon^m \langle (W_1 \cdots W_k)^m \rangle = \epsilon^m \langle W_1^m \rangle \cdots \langle W_k^m \rangle$

but the W_is are all distributed in the same way and $l = \alpha^k L \Rightarrow k = \log_\alpha\left(\frac{l}{L}\right)$, so $\langle \epsilon_l^m \rangle = \epsilon^m \langle W^m \rangle^k = \epsilon^m \alpha^{k \log_\alpha \langle W^m \rangle} = \epsilon^m \left(\frac{l}{L}\right)^{\log_\alpha \langle W^m \rangle}$.

$\therefore \langle \epsilon_l^m \rangle = \epsilon^m \left(\frac{l}{L}\right)^{\mu_m}$, where $\mu_m = \log_\alpha \langle W^m \rangle = \frac{\ln \langle W^m \rangle}{\ln \alpha}$

Applying this model to K41, $W = 1$ and $\mu_m = 0$.

To model intermittency in K62, we need to figure out how W is distributed.

First, our choice of α must not influence the result, so replacing α by α^p, $p \in \mathbb{Z}^+$, shouldn't change the result $\Rightarrow W_1$ and $W_1 \cdots W_p$ must have the same distribution. If we define $w_i = \ln W_i$, then $\sum_{i=1}^{p} w_i$ must have the same distribution as w_i, because $W_1 \cdots W_p = e^{\sum_{i=1}^{p} w_i}$ has the same distribution as W_i.

i) Lognormal Model

An obvious candidate distribution is the Gaussian, $f(w) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(w - \langle w \rangle)^2}{2\sigma^2}}$. Then, $\sum_{i=1}^{p} w_i$ is also Gaussian with mean $p\langle w \rangle$ and variance $p\sigma^2$. $\therefore \epsilon_l$ is lognormal (Obukhov 1962)

$\langle W^m \rangle = \langle e^{mw} \rangle = \int dw\, e^{mw} f(w) = \frac{1}{\sqrt{2\pi\sigma^2}} \int dw\, e^{mw - \frac{(w - \langle w \rangle)^2}{2\sigma^2}} = e^{m\langle w \rangle + \frac{\sigma^2 m^2}{2}}$.

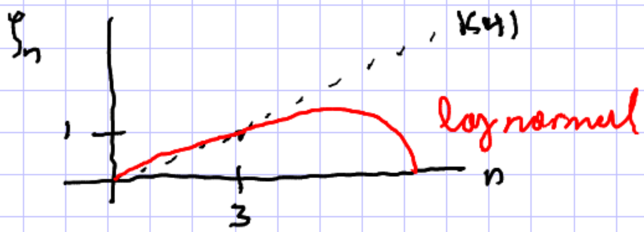
We know that $\langle W \rangle = 1$, so $\langle w \rangle + \frac{\sigma^2}{2} = 0$

$$\Rightarrow \langle W^m \rangle = e^{\langle w \rangle m(1-m)}$$

Recall that $\mu_m = \frac{\ln\langle W^m\rangle}{\ln\alpha} = \underline{\frac{\langle w\rangle}{\ln\alpha}}\, m(1-m)$

free parameter $= \frac{\mu}{2}$

$\therefore\ \zeta_n = \frac{n}{3} + \mu_{n/3} = \frac{n}{3} + \frac{\mu}{18} n(3-n)$



This is not good, since we know ζ_n is monotonic

ii) Log Poisson (She-Lévêque 1994)

Another candidate distribution is the Poisson distribution: $f(q) = \text{Probability}(X=q) = \frac{\lambda^q e^{-\lambda}}{q!}$,

where $q = 0,1,2,\ldots$, $\lambda\in\mathbb{R}$, $\lambda>0$, $\lambda = \langle q\rangle = \langle q^2\rangle^{1/2}$

and X is a discrete random variable.

This distribution $\Rightarrow$ $w = q\ln\beta + w_0$ $\quad (W = e^{w_0}\beta^q)$

where β and w_0 are parameters.

ϵ_ℓ is now log Poisson

$$\langle W^m\rangle = \langle e^{mw}\rangle = \sum_{q=0}^{\infty} e^{mq\ln\beta + mw_0 - \lambda}\left(\frac{\lambda^q}{q!}\right)$$

$$= e^{mw_0-\lambda}\underbrace{\sum_{q=0}^{\infty}\frac{(\lambda\beta^m)^q}{q!}}_{e^{\lambda\beta^m}} = e^{mw_0-\lambda(1-\beta^m)}$$

$\langle W\rangle = 1 \Rightarrow w_0 - \lambda(1-\beta^m) = 0$ or $\lambda = \frac{w_0}{1-\beta}$

$\therefore\ \langle W^m\rangle = e^{w_0\left(m - \frac{1-\beta^m}{1-\beta}\right)}$ and

$$\mu_m = \frac{\ln\langle W^m\rangle}{\ln\alpha} = \underbrace{\frac{w_0}{\ln\alpha}}_{-X}\left(m - \frac{1-\beta^m}{1-\beta}\right) \Rightarrow$$

$$\zeta_n = \frac{n}{3} + \mu_{n/3} = \frac{n}{3}(1-X) + X\,\frac{1-\beta^{n/3}}{1-\beta}$$

X and β are both parameters to determine

Note that ξ_n is monotonic for $\beta < 1$

Let's now see if we can figure out values for β and x:

Well, $\langle \epsilon_e^m \rangle = \epsilon^m \left(\frac{\ell}{L}\right)^{-x\left(m - \frac{1-\beta^m}{1-\beta}\right)}$

If we consider $m \to \infty$, then

$$\langle \epsilon_e^m \rangle \sim \epsilon^m \left(\frac{\ell}{L}\right)^{-xm + \frac{x}{1-\beta}} \quad (1)$$

Remember that the higher the order of the moment considered, the more focused the quantity is on the extreme, i.e. singular, structures. So,

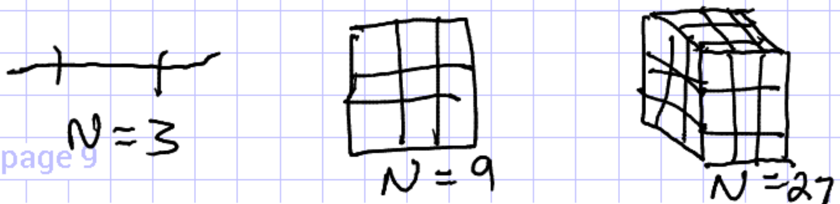
$$\langle \epsilon_e^m \rangle \underset{m\to\infty}{\sim} [\epsilon_e^{(\infty)}] \, \chi_e$$

$\epsilon_e^{(\infty)}$: power dissipated in these structures

χ_e: volume filling fraction of the structures

χ_e must thus be of the form $\chi_e \sim \left(\frac{\ell}{L}\right)^{d-D}$, where d is the dimension of the embedding space considered ($d = 3$) and D is the dimension of the most singular structures. D is the fractal dimension and $d - D$ is called the codimension, C

Aside on fractal dimension: It is a ratio that provides a statistical index of complexity describing how detail changes with scale. The easiest intuitive picture is this: $N \propto \epsilon^{-D}$, where N is the # of new components, ϵ is the scaling factor, and D is the fractal dimension. So, consider dividing a line, a rectangle, and a box by thirds, $\epsilon = 1/3$



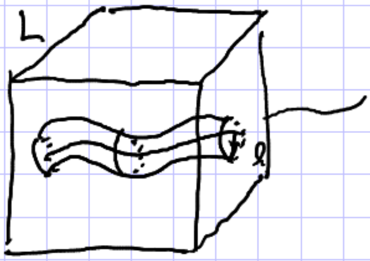
So, $D = 1$ for the line
$= 2$ for the rectangle
$= 3$ for the box

Note that D need not be an integer.

Back to original discussion

$\chi_\ell \sim \left(\frac{\ell}{L}\right)^{d-D}$ follows from it describing the probability for a ball of radius ℓ to intersect with a dissipative structure for $\ell \to 0$ (⊛ Also see alternate derivation). In $d=3$:

Filaments

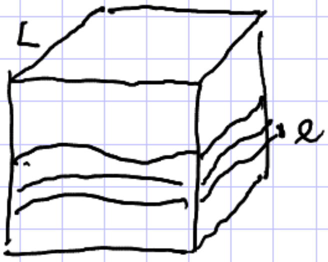


Volume $\sim \ell^2 L \Rightarrow$

$$\chi_\ell \sim \frac{\ell^2 L}{L^3} \sim \left(\frac{\ell}{L}\right)^{2 = d-D} \Rightarrow D = 1$$

$C = 2$

Sheets

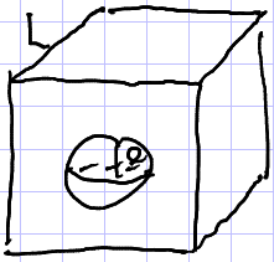


Volume $\sim \ell L^2 \Rightarrow$

$$\chi_\ell \sim \frac{\ell L^2}{L^3} \sim \left(\frac{\ell}{L}\right)^1 \Rightarrow D = 2$$

$C = 1$

Spheres



Volume $\sim \ell^3 \Rightarrow$

$$\chi_\ell \sim \left(\frac{\ell}{L}\right)^3 \Rightarrow D = 0$$

$C = 3$

Note that D need not be an integer

$$\therefore \langle \epsilon_\ell^m \rangle \sim [\epsilon_\ell^{(\infty)}]^m \left(\frac{\ell}{L}\right)^C \text{ and } ① \Rightarrow$$

$$\langle \epsilon_\ell^m \rangle \sim \left[\epsilon \left(\frac{\ell}{L}\right)^{-x}\right]^m \left(\frac{\ell}{L}\right)^{\frac{x}{1-\beta}}$$

$$\therefore \epsilon_\ell^{(\infty)} \sim \epsilon \left(\frac{\ell}{L}\right)^{-x} \text{ and } \frac{x}{1-\beta} = C \text{ and}$$

$$\zeta_n = (1-x)\frac{n}{3} + C\left[1 - \left(1 - \frac{x}{C}\right)^{n/3}\right]$$

She-Lévêque assumed that most of the energy in the cascade is dissipated in these singular structures and that the cascade time obeys the K41 scaling

$$\epsilon_\ell^{(\infty)} \sim \frac{u_0^2}{\tau_\ell} \sim u_0^2\, \epsilon^{1/3} \ell^{-2/3} \Rightarrow x = 2/3$$

We know empirically that the primary dissipative structures are filaments, so $C=2$

$\therefore \quad \zeta_n = \frac{n}{9} + 2[1-(\frac{2}{3})^{n/3}]$ Log-Poisson model

This fits the experimental data well

D Extended Self-Similarity (Benzi et al 1993)

measuring high order S_n can be difficult for only moderate Re. So, Benzi et al 1993 created a trick: Since $S_3(r) \sim r$, why not measure $S_n(r)$ vs $S_3(r)$ rather than $S_n(r)$ vs r?

this approach $\Rightarrow S_n(r) \sim r^{\zeta_n} \sim S_3(r)^{\zeta_n}$.

$$\therefore \quad \zeta_n = \frac{\partial \ln S_n(r)}{\partial \ln S_3(r)}$$

This works because the "noise" in the structure functions is correlated across all orders. So, it effectively cancels using this method. The method is called extended self-similarity because it cures the lack of asymptocity present due to the low Reynold's number by observing that the functional form of the structure functions is the same at the beginning of the dissipation range.

⊛ Alternate derivation of χ_ℓ

$$\chi_\ell = \frac{N\ell^d}{L^d} \sim \left(\frac{L}{\ell}\right)^D \left(\frac{\ell}{L}\right)^d = \left(\frac{\ell}{L}\right)^{d-D}$$

, where N is the # of eddies of size ℓ and I used $N \sim \epsilon^{-D}$ with ϵ = the scaling factor $= \frac{\ell}{L}$