Motion of charged particles in electromagnetic fields

Libby Tolman

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Abstract

These are notes for the Thursday, June 10th, 10 am EDT lecture at the 2021 NSF/APS-DPP GPAP Summer School. They borrow from lecture notes available at `https://www.astro.princeton.edu/~kunz/Site/AST521/AST521_lecture_notes_Kunz.pdf`, material in Chen's *Introduction to Plasma Physics and Controlled Fusion*, available at `https://link.springer.com/book/10.1007/978-3-319-22309-4`, and material in Goldston and Rutherford's *Introduction to Plasma Physics* (no online version that I'm aware of). Please send comments/corrections to tolman@ias.edu.

1 Introduction and context

So far, you have learned how to study plasmas as though they were a conducting fluid. This morning, we'll start to consider how individual charged particles (electrons and ions) move in electromagnetic fields, which will lead later on in the day to a discussion of kinetic theory, which treats a plasma as a collection of particles all moving separately under the influence of electromagnetic fields. Virtually all of this lecture is a consequence of one equation:

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right), \tag{1}$$

which determines the motion of a particle of charge q and mass m in an electric field $\vec{E}$ and a magnetic field $\vec{B}$. However, the consequences of this equation are surprisingly subtle.

2 Uniform electric and magnetic fields

2.1 Gyromotion

We start by considering how a particle moves in just a magnetic field, i.e., under

$$m\frac{d\vec{v}}{dt} = q\left(\frac{\vec{v}\times\vec{B}}{c}\right). \tag{2}$$

With $\vec{B}$ in the $\hat{z}$ direction, we have

$$m\dot{v}_x = qBv_y/c,\ m\dot{v}_y = -qBv_x/c,\ m\dot{v}_z = 0. \tag{3}$$

Taking an additional derivative gives

$$\ddot{v}_x = \frac{qB}{mc}\dot{v}_y = -\left(\frac{qB}{mc}\right)^2 v_x \tag{4}$$

and

$$\ddot{v}_y = -\frac{qB}{mc}\dot{v}_x = -\left(\frac{qB}{mc}\right)^2 v_y \tag{5}$$

Let's define the cyclotron frequency,

$$\omega_c \equiv \frac{|q|\,B}{mc} \tag{6}$$

These equations are solved by

$$v_x = v_\perp \cos\left(\omega_c t\right), \tag{7}$$

$$v_y = -\frac{|q|}{q} v_\perp \sin\left(\omega_c t\right), \tag{8}$$

which can be integrated to give

$$x\left(t\right) = x_0 + \frac{v_\perp}{\omega_c}\sin\left(\omega_c t\right) \tag{9}$$

$$y\left(t\right) = y_0 + \frac{|q|}{q}\frac{v_\perp}{\omega_c}\cos\left(\omega_c t\right), \tag{10}$$

from which we can define the Larmor radius

$$r_l \equiv \frac{v_\perp}{\omega_c} = \frac{mcv_\perp}{|q|\,B}. \tag{11}$$

These equations describe gyromotion, which is illustrated in Figure 1.

2.2 $\vec{E} \times \vec{B}$ drift

Now, let's consider the introduction of a uniform, time-independent electric field, such that our equation of motion is

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right). \tag{12}$$

Let's consider how a particle moves in this set of fields. To do this, I'm going to introduce a new variable that won't make sense at first:

$$\vec{u} \equiv \vec{v} - c\left(\vec{E}\times\vec{B}\right)/B^2. \tag{13}$$

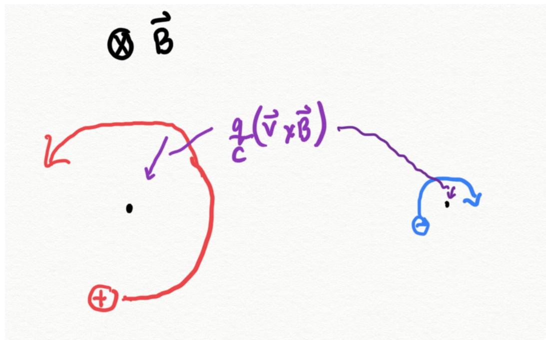

Figure 1: Gyromotion of an electron and an ion.

This is the velocity a particle would have if looked at in a frame moving with $c\left(\vec{E}\times\vec{B}\right)/B^2$.

If I replace $\vec{v}$ in (15) with $\vec{u}+c\left(\vec{E}\times\vec{B}\right)/B^2$, i.e.,

$$m\frac{d\vec{u}}{dt} = q\left[\vec{E}+\frac{\vec{u}\times\vec{B}}{c}+\frac{\left(\vec{E}\times\vec{B}\right)\times\vec{B}}{B^2}\right], \tag{14}$$

and apply vector identities, I arrive at

$$m\frac{d\vec{u}}{dt} = q\left[\hat{b}\left(\vec{E}\cdot\hat{b}\right)+\frac{\vec{u}\times\vec{B}}{c}\right]. \tag{15}$$

Let's consider what this equation means. Its component parallel to the magnetic field is just

$$m\frac{du_\parallel}{dt} = qE_\parallel, \tag{16}$$

which is normal motion parallel to an electric field. The perpendicular component is

$$m\frac{d\vec{u}_\perp}{dt} = q\left[\frac{\vec{u}_\perp\times\vec{B}}{c}\right], \tag{17}$$

which gives the same gyromotion that we solved for in 2.1. This means that in a frame moving with the velocity

$$\vec{v}_{E\times B} = c\left(\vec{E}\times\vec{B}\right)/B^2, \tag{18}$$

the perpendicular dynamics is just gyromotion. This means that the overall particle trajectory is gyration plus a drift $\vec{v}_{E\times B}$. Let's take a look at this, in Figure 2.

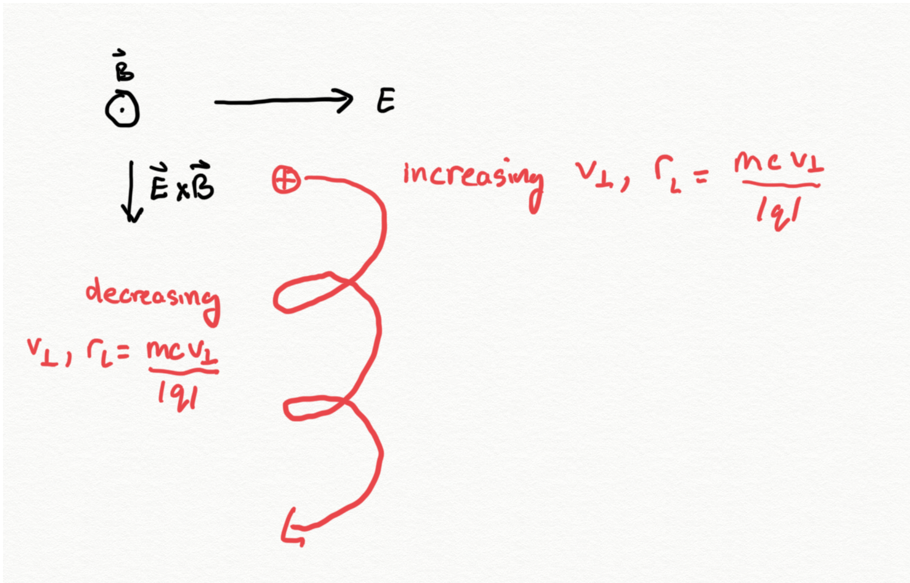

Figure 2: $\vec{E} \times \vec{B}$ drift.

3 Non-uniform electric and magnetic fields

3.1 Grad-B drift

Another particle drift occurs in a magnetic field that varies in space perpendicular to its direction. The math to derive this drift is slightly more involved, so let's just look at it physically. Consider Figure 3.

Calculation of the drift (using perturbation theory; a good explanation can be found in Goldston and Rutherford Ch. 3) gives that its value is

$$\vec{v}_{\nabla B} = \pm \frac{v_\perp r_l}{2} \frac{\vec{B} \times \nabla B}{B^2}, \tag{19}$$

with the $\pm$ corresponding to the sign of the charge.

3.2 Magnetic mirrors

Finally, let's consider particle motion in a magnetic field that varies along its direction. The setup is in Figure 4, where the magnetic field strength varies along $\hat{z}$ (I'll be using cylindrical coordinates in this section). First, note that although the magnetic field is on the left side just along $\hat{z}$, it will develop a radial component as it gains strength, which can be found from

$$\nabla \cdot \vec{B} = \frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{\partial B_z}{\partial z} = 0. \tag{20}$$

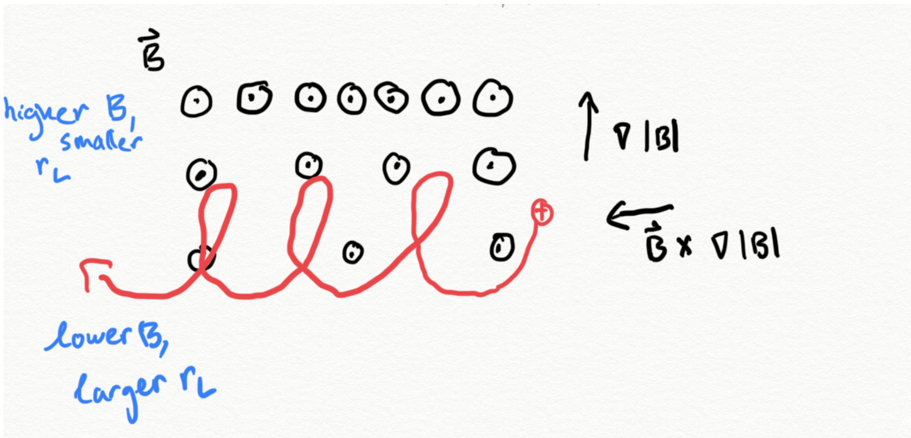

Figure 3: Grad B drift.

Assuming approximately constant $\partial B_z/\partial z$, we find

$$rB_r = -\int_0^r r\frac{\partial B_z}{\partial z}dr \approx -\frac{r^2}{2}\frac{\partial B_z}{\partial z}, \tag{21}$$

$$B_r \approx -\frac{r}{2}\frac{\partial B_z}{\partial z}, \tag{22}$$

Let's examine the z component of component of the force caused by this radial magnetic field:

$$F_z = -\frac{q}{c}v_\theta B_r. \tag{23}$$

Consider also for simplicity a particle with a guiding center at the center ($r = 0$) of the mirror, such that $v_\theta = \mp v_\perp$ is constant during a gyration. Then, we find the average force

$$\bar{F}_z = \mp\frac{1}{2c}qv_\perp r_l\frac{\partial B_z}{\partial z} = -\frac{1}{2}\frac{mv_\perp^2}{B}\frac{\partial B_z}{\partial z} \equiv -\mu\frac{\partial B_z}{\partial z}, \tag{24}$$

where we have defined the magnetic moment,

$$\mu \equiv \frac{1}{2}mv_\perp^2/B. \tag{25}$$

This is a specific form of the force on a diamagnetic particle,

$$\vec{F}_\| = -\mu\partial B/\partial\vec{s}, \tag{26}$$

with $d\vec{s}$ a line element along $\vec{B}$. Let's consider the equation of motion along $\vec{B}$:

$$m\frac{dv_\|}{dt} = -\mu\frac{\partial B}{\partial s}. \tag{27}$$

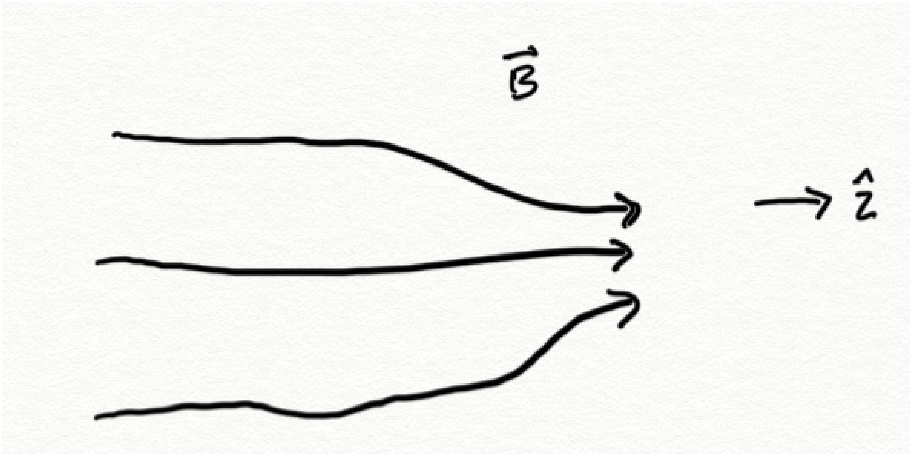

Figure 4: Magnetic mirror setup.

Multiplying by $v_\parallel$ gives

$$mv_\parallel \frac{dv_\parallel}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\frac{\partial B}{\partial s}\frac{ds}{dt} = -\mu\frac{dB}{dt}. \tag{28}$$

(Note that the right side is the time variation seen by the particle, the actual field doesn't change with time.) Since the particle's energy is constant, I can write

$$\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = -\frac{d}{dt}\left(\mu B\right) = -\mu\frac{dB}{dt}. \tag{29}$$

The last two expressions can be restated as

$$\frac{d\mu}{dt} = 0, \tag{30}$$

which means that the magnetic moment is conserved throughout the particle motion. (It is called an "adiabatic invariant.") This means that as a particle approaches a region of higher magnetic field, its perpendicular velocity will increase to keep μ constant. This will decrease the particle's parallel velocity (so that energy is constant), causing the particle to "mirror" and reverse direction. This is shown schematically in Figure 5.

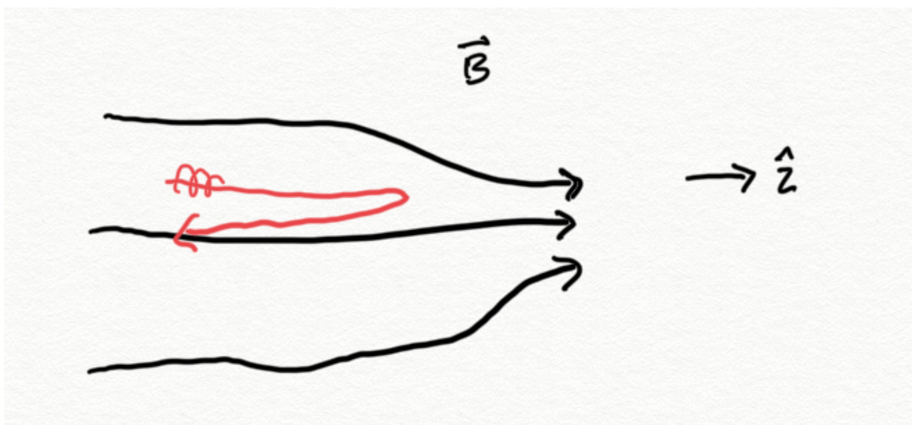

Figure 5: Magnetic mirror in action.