MHD Waves: Alfvén, fast, and slow modes (GPAP school notes)

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Abstract

These are notes for the Tuesday, June 8th, 10 am EDT lecture at the 2021 NSF/APS-DPP GPAP Summer School. They borrow heavily from lecture notes available at `https://www.astro.princeton.edu/~kunz/Site/AST521/AST521_lecture_notes_Kunz.pdf`. Please send comments/corrections to tolman@ias.edu.

1 MHD equations (review from yesterday)

Yesterday, Prof. Brown introduced the MHD equations. These are (in ideal form):

- the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \tag{1}$$

- the momentum equation,

$$\rho \left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right) \vec{u} = \frac{\vec{j} \times \vec{B}}{c} - \nabla p \tag{2}$$

- Ohm's law,[1]

$$\vec{E} + \frac{\vec{u} \times \vec{B}}{c} = 0 \tag{3}$$

- Ampère's law,

$$\vec{j} = \frac{c}{4\pi} \nabla \times \vec{B}, \tag{4}$$

- Faraday's law,

$$\frac{\partial \vec{B}}{\partial t} = -c \nabla \times \vec{E} \tag{5}$$

[1] By combining with Faraday's law, this can also be expressed via the induction equation, $\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{u} \times \vec{B} \right)$.

Quantity	Definition
ρ	mass density
$\vec{u}$	plasma velocity
$\vec{j}$	current density
$\vec{B}$	magnetic field
c	speed of light
p	plasma pressure
γ	ratio of specific heats (i.e., 5/3)

Table 1: Definitions of parameters in the MHD equations.

- divergence constraint,

$$\nabla \cdot \vec{B} = 0, \tag{6}$$

- and the adiabatic energy equation,

$$\left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right)\left(\frac{p}{\rho^{\gamma}}\right) = 0. \tag{7}$$

The definitions of the parameters in these equations are given in Table 1. Today, we're going to explore waves that these equations allow. For this purpose, it will be useful to rewrite (2) using (4) as

$$\begin{aligned}\rho\left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right)\vec{u} = \frac{\vec{j} \times \vec{B}}{c} - \nabla p &= \frac{1}{4\pi}\left(\nabla \times \vec{B}\right) \times \vec{B} - \nabla p \\ &= -\nabla p - \frac{1}{8\pi}\nabla B^2 + \frac{1}{4\pi}\vec{B} \cdot \nabla \vec{B} \\ &= -\nabla p - \frac{1}{8\pi}\nabla_{\perp} B^2 + \frac{B^2}{4\pi}\vec{b} \cdot \nabla \vec{b}. \end{aligned} \tag{8}$$

Here, $B^2/(8\pi)$ is magnetic pressure, and represents the tendency of the magnetic field to want to be the same strength everywhere. $\frac{B^2}{4\pi}\vec{b}\cdot\nabla\vec{b}$ is a curvature/tension term that represents the desire of magnetic field lines to be straight. These forces are illustrated in Figure 1.

2 Linearization

Let's consider the $\hat{z}$ direction to be along the magnetic field. Waves can exist at an arbitrary angle to the magnetic field, with a wave vector

$$\vec{k} = k_{\parallel}\hat{z} + \vec{k}_{\perp}, \tag{9}$$

which is indicated in Figure 2. Then, we can consider small perturbations of

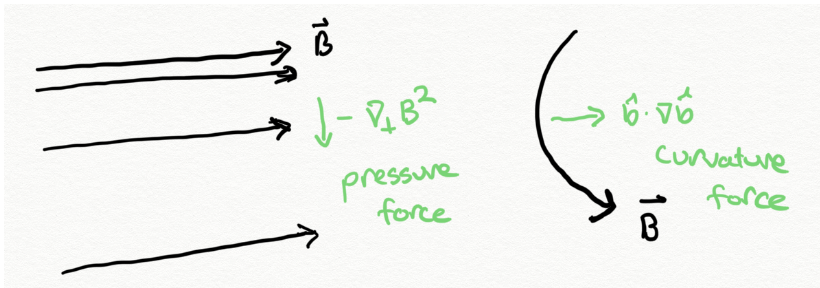

Figure 1: Magnetic forces.

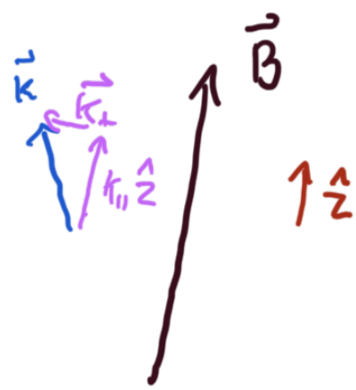

Figure 2: The magnetic field and wave vector setup.

the various plasma quantities around the equilibrium:

$$\rho = \rho_0 + \rho_1 e^{i\vec{k}\cdot\vec{r}-i\omega t}, \tag{10}$$

$$\vec{B} = B_0\hat{z} + \vec{B}_1 e^{i\vec{k}\cdot\vec{r}-i\omega t} \tag{11}$$

$$\vec{u} = 0 + \vec{u}_1 e^{i\vec{k}\cdot\vec{r}-i\omega t}, \tag{12}$$

and

$$p = p_0 + p_1 e^{i\vec{k}\cdot\vec{r}-i\omega t}. \tag{13}$$

The perturbations are small enough that any nonlinearities (e.g., $p_1 p_1$) can be dropped. Here, we follow the normal convention that to get observed quantities, you take the real component, i.e.

$$e^{i\vec{k}\cdot\vec{r}-i\omega t} \to \cos\left(\vec{k}\cdot\vec{r}-\omega t\right) \tag{14}$$

$$ie^{i\vec{k}\cdot\vec{r}-i\omega t} \to -\sin\left(\vec{k}\cdot\vec{r}-\omega t\right). \tag{15}$$

3 Alfvén waves

Let's start with a wave that has just $\vec{k} = k_\parallel \hat{z}$. From continuity, (1), we get

$$\partial_t\left(\rho_0 + \rho_1 e^{ik_\parallel z - i\omega t}\right) + \nabla\cdot\left[\left(\rho_0 + \rho_1 e^{ik_\parallel z - i\omega t}\right)\left(0 + \vec{u}_1 e^{ik_\parallel z - i\omega t}\right)\right] = 0. \tag{16}$$

Linearizing gives

$$\partial_t\left(\rho_1 e^{ik_\parallel z - i\omega t}\right) + \rho_0 \nabla\cdot\left[\left(\vec{u}_1 e^{ik_\parallel z - i\omega t}\right)\right] = 0, \tag{17}$$

or

$$-i\omega\frac{\rho_1}{\rho_0} + ik_\parallel u_{1,\parallel} = 0. \tag{18}$$

Similar manipulation on (8) gives

$$-i\omega\vec{u}_1 = -\frac{ik_\parallel \hat{z}}{\rho_0}\left(p_1 + \frac{B_0 B_{1\parallel}}{4\pi}\right) + \frac{ik_\parallel B_0}{4\pi\rho_0}\vec{B}_1, \tag{19}$$

and using the induction equation, (Ohm's law + Faraday's law), $\frac{\partial \vec{B}}{\partial t} = \nabla\times\left(\vec{u}\times\vec{B}\right)$,

$$-i\omega\frac{\vec{B}_1}{B_0} = ik_\parallel\vec{u}_1 - \hat{z}ik_\parallel u_{1,\parallel}. \tag{20}$$

Note that this can be used to give

$$B_{1\parallel} = 0, \tag{21}$$

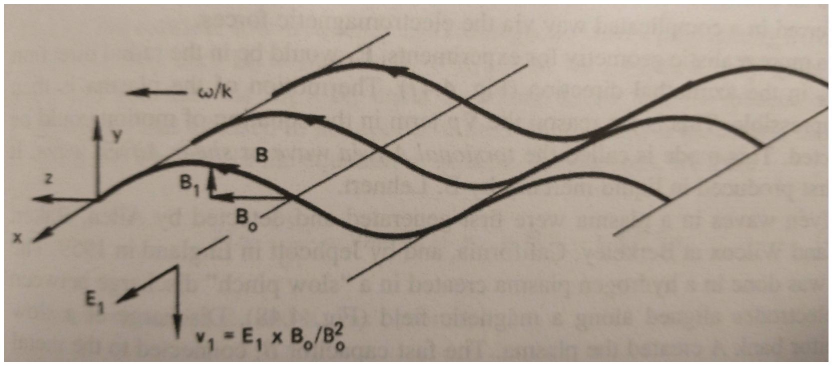

Figure 3: An Alfvén wave (from *Introduction to Plasma Physics and Controlled Fusion*, by F.F. Chen.)

which agrees with

$$\nabla \cdot \left(\vec{B}_1 e^{ik_\parallel z - i\omega t} \right) = ik_\parallel B_{1\parallel} e^{ik_\parallel z - i\omega t} = 0. \tag{22}$$

Let's consider the perpendicular waves in this system:

$$\rho_1 = 0, \tag{23}$$

$$-i\omega \vec{u}_{1\perp} = \frac{ik_\parallel B_0}{4\pi\rho_0} \vec{B}_{1\perp}, \tag{24}$$

and

$$-i\omega \frac{\vec{B}_{1_\perp}}{B_0} = ik_\parallel \vec{u}_{1\perp}. \tag{25}$$

Using (24) to solve for $\vec{u}_{1\perp}$ and substituting into (25) gives

$$\omega = \pm k_\parallel \frac{B_0}{\sqrt{4\pi\rho_0}} \equiv \pm k_\parallel v_A, \tag{26}$$

where we have defined the Alfvén speed

$$v_A \equiv \frac{B_0}{\sqrt{4\pi\rho_0}}. \tag{27}$$

The Alfvén wave is pictured in Figure 3. You can think of it like plucking the strings on a guitar. The tension force of the magnetic field pulls the strings back, but they are slowed down by the weight of the plasma. You can see that the phase velocity of the wave increases with magnetic field strength and decreases with plasma mass.

The discovery of Alfvén waves received the 1970 Nobel prize. They are important across plasma physics. In the sun they heat the solar corona. In terrestrial plasma experiments, they can move particles around in undesired ways, an effect that I studied in my PhD thesis.

4 Slow and fast waves

We can also linearize with a more general wave vector,

$$\vec{k} = k_\parallel \hat{z} + \vec{k}_\perp, \tag{28}$$

in the same way that we did before. This is a somewhat unpleasant process which can be found on page 63 of `https://www.astro.princeton.edu/~kunz/Site/AST521/AST521_lecture_notes_Kunz.pdf`. Let's skip to the end result:

$$\left(\omega^2 - k_\parallel^2 v_A^2\right)\left[\omega^2 - k_\parallel^2 v_A^2 - k_\perp^2 v_A^2 \left(\frac{\omega^2}{\omega^2 - k^2 c_s^2}\right)\right] = 0. \tag{29}$$

Here, we have defined a sound speed,

$$c_s \equiv \sqrt{\frac{\gamma p_0}{\rho_0}}. \tag{30}$$

(This is the speed at which compressions and rarefactions of the plasma move along the magnetic field without moving the field.) The first factor is just our familiar Alfvén wave. The second factor has two new waves:

$$\omega^2 = \frac{k^2\left(c_s^2 + v_A^2\right)}{2} \pm \sqrt{\frac{k^4\left(c_s^2 + v_A^2\right)^2}{4} - k_\parallel^2 v_A^2 k^2 c_s^2}. \tag{31}$$

These are called "magnetosonic waves." The fast wave is given by the + solution and the slow wave by the − solution.

Let's look into what these waves are like in one limit.

For the fast wave, consider $k_\parallel = 0$. Then, we have

$$\omega^2 = k^2\left(c_s^2 + v_A^2\right). \tag{32}$$

This wave is illustrated in Figure 4.

For the slow wave, we can consider $k_\parallel \ll k_\perp$. Then, the dispersion relation becomes

$$\omega^2 \approx k_\parallel^2 v_A^2 \left(\frac{c_s^2}{c_s^2 + v_A^2}\right). \tag{33}$$

This wave is shown in Figure 5.

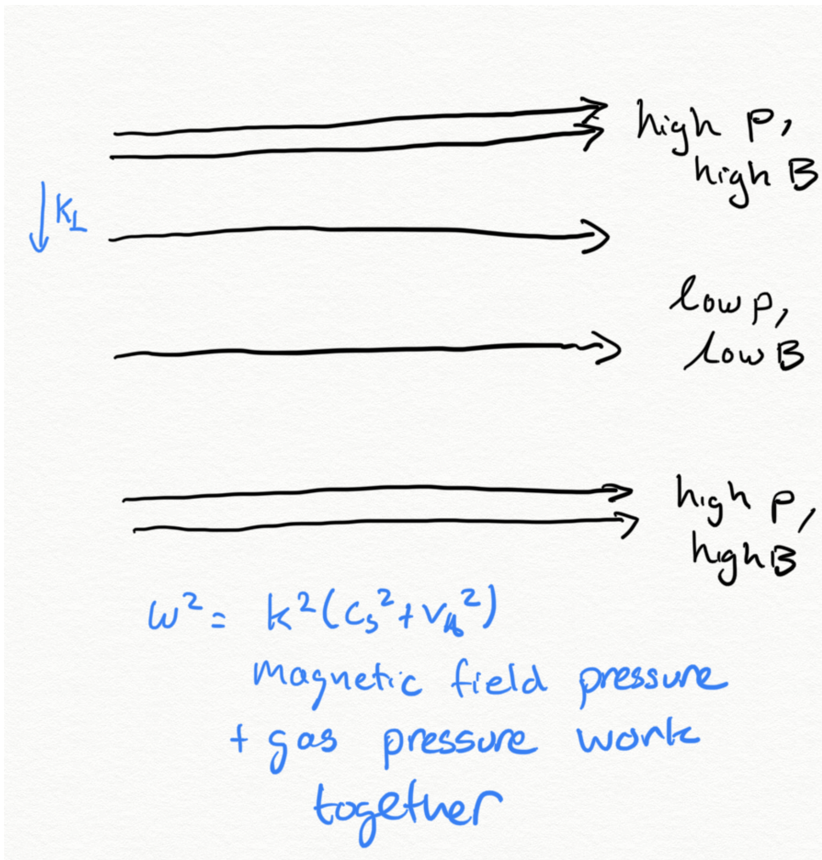

Figure 4: The fast wave with $k_\parallel = 0$.

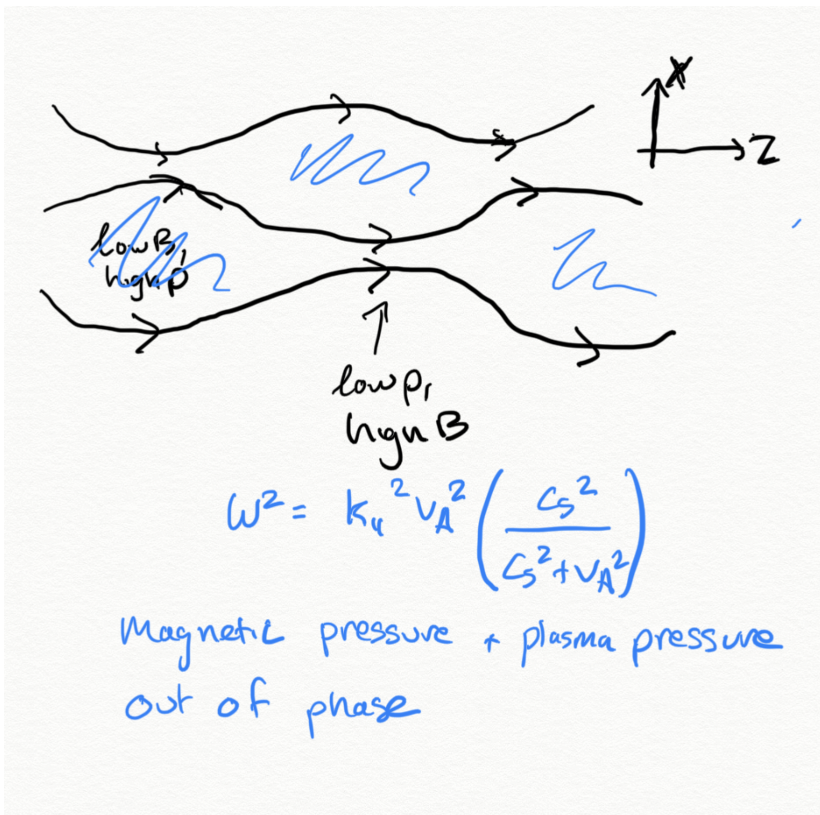

Figure 5: The slow wave with $k_\parallel \ll k_\perp$.