II An Introduction to Plasma Turbulence

Now that we have visited some of the important neutral fluid turbulence topics, we can now re-examine some of the same concepts for plasmas. Since we are now going to couple the velocity field to E and B, things are obviously going to get messier in some ways, but it turns out that it also becomes simpler in others.

Recall that in hydro turbulence

(I) Scale invariance and

(II) Locality of energy transfer

$$\Rightarrow \quad \epsilon \sim \frac{\delta u_\ell^2}{\tau_\ell} \sim \text{const.}$$

Further, there was only one time scale in the system, the eddy-turnover time

$$\tau_\ell \sim \frac{\ell}{\delta u_\ell} \Rightarrow \epsilon \sim \frac{\delta u_\ell^3}{\ell} \sim \text{const} \quad \therefore \delta u_\ell \sim (\epsilon \ell)^{1/3}$$

We had the Kolmogorov 41 5/3 energy spectrum from simple dimensional analysis.

Now, let's look at the incompressible MHD eqns. We'll stick with the incompressible assumption because it makes things simpler, but it also turns out that Alfvon waves (which are incompressible) are the primary component of turbulence.

$$\rho_0(\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u}) = -\vec{\nabla}P + \frac{\vec{B}\cdot\nabla\vec{B}}{4\pi} + \nu\nabla^2\vec{u}\rho_0 \quad (1)$$

$$\partial_t \vec{B} + \vec{u}\cdot\nabla\vec{B} = \vec{B}\cdot\nabla\vec{u} + \underbrace{\eta}_{\text{resistivity}}\nabla^2\vec{B} \quad (2)$$

where $P = p + \frac{B^2}{8\pi}$ is the total pressure

If we assme we have a mean magnetic field, $\vec{B_0} = B_0\hat{z}$, we immediately break the isotropy assumption of hydro because, unlike a mean flow, a mean B_0 cannot be transformed away and introduces linear Alfven waves. So, we now have an inherently anisotropic system w/ Alfven waves III

Let's re-write eqns ① and ② in a more compact form by adding and subtracting them to obtain the Elsasser (1950) eqns

$$③ \quad \partial_t \vec{z}^{\pm} \mp (\vec{V}_A \cdot \nabla)\vec{z}^{\pm} + (\vec{z}^{\mp}\cdot\nabla)\vec{z}^{\pm} = -\frac{\vec{\nabla} p}{\rho_0} + \frac{\nu+\eta}{2}\nabla^2 \vec{z}^{\pm} + \frac{\nu-\eta}{2}\nabla^2 \vec{z}^{\mp}$$

where $\vec{z}^{\pm} = \vec{u} \pm \frac{\delta \vec{B}}{\sqrt{4\pi\rho_0}}$; $\vec{V}_A = V_A \hat{z}$; $\delta\vec{B} = \vec{B} - \vec{B}_0$

A few important notes about ~~these eqns~~

1) $\vec{z}^- = 0$ and $\vec{z}^+ = f(x, y, z+V_A t)$ or
$\vec{z}^+ = 0$ and $\vec{z}^- = f(x, y, z-V_A t)$
are exact solutions. They represent waves travelling down B_0 ($\vec{z}^+$) or up B_0 ($\vec{z}^-$)

2) The system supports two linear wave modes, both satisfying $\omega^2 = k_\parallel^2 V_A^2$. These are the Alfven waves with polarization in the $\hat{z}\times\hat{k}$ direction and $\hat{k}\times(\hat{z}\times\hat{k})$ for the pseudo-Alfven waves, which are the incompressible limit of magnetosonic slow modes. Fast waves are ordered out of the system due to the incompressibility assumption, ie, $c_s \to \infty$. It turns out that the slow modes are passive (will show this in a later lecture), so we will focus only on the Alfven modes.

3) The system is closed. Taking the divergence of ③

$$\frac{1}{\rho_0}\nabla^2 p = -\nabla\cdot(\vec{z}^{\mp}\cdot\nabla\vec{z}^{\pm})$$

4) The nonlinear term, $\vec{z}^- \cdot \nabla \vec{z}^+$ requires oppositely propagating Alfven waves. Further, if we just look at the

behavior of the dot product subject to the constraints above and Fourier transformation

$$ik^+ z^-_{k^-} z^+_{k^+}\left[(\hat{z}\times\hat{k}^-)\cdot\hat{k}^+\right](\hat{z}\times\hat{k}^+) = ik^+ z^-_{k^-} z^+_{k^+}(\hat{z}\times\hat{k}^+)\left[\hat{z}\cdot(\hat{k}^-\times\hat{k}^+)\right]$$

So, the nonlinear term also requires that $\vec{z}^+$ and $\vec{z}^-$ have nonzero relative polarization.

Since we now know we are dealing with Alfvenic fluctuations, we know everything is in an Alfvenic state, $\delta u_\ell \sim \delta B_\ell$ scale-by-scale (IV) $\Rightarrow$ same spectra for u and B. Given (II) $\rightarrow$ (IV), can we now construct the energy spectra for Alfvenic turbulence using the same dimensional arguments as K41?

$\epsilon_\ell \sim \frac{\delta u_\ell^2}{\tau_\ell} \sim$ const is still ok

but we now have 2 choices for τ_ℓ

- eddy turn over time $\tau_{eddy} \sim \frac{\ell_\perp}{\delta u_\ell}$ (Nonlinear time)
- Alfven time $\tau_A \sim \frac{\ell_\parallel}{v_A}$

So, which one do we choose and why?

Iroshnikov (1964) - Kraichnan (1965) Theory

To derive an energy spectrum, IK further assumed that the turbulence is weak:

The nonlinear term $|\vec{z}^\mp\cdot\nabla\vec{z}^\pm| \ll |v_A\nabla_\parallel\vec{z}^\pm|$ linear term (V)

This ratio $\frac{|z^\mp\cdot\nabla z^\pm|}{|v_A\nabla_\parallel z^\pm|} \sim \frac{z^\mp k_\perp}{v_A k_\parallel} \sim \frac{\delta u_\perp k_\perp}{v_A k_\parallel} = \frac{\tau_A}{\tau_{eddy}} =: \chi$, the nonlinearity parameter

$\chi \ll 1 \Rightarrow$ weak turbulence, linear term dominates

$\chi \gtrsim 1 \Rightarrow$ strong turbulence

This feature of turbulence was absent the hydro systems we examined, but it exists when linear terms come into play, eg, Mach $\gtrsim 1$ flows, gravity waves, shallow water waves, etc.

When $\chi \ll 1$, each wave-wave interaction only decorrelates the wave a little. So,

Crossing / interaction time : $\Delta t \sim \frac{l_\parallel}{v_A} \sim \tau_A$

Change in amplitude: $\partial_t \Delta \delta u \sim \delta u \cdot \nabla \delta u$

$$\Rightarrow \Delta \delta u_l \sim \frac{\delta u_l^2}{l_\perp} \Delta t \sim \delta u_l \frac{\delta u_l}{l_\perp} \frac{l_\parallel}{v_A} \sim \delta u_l \chi$$

The "kicks" to the amplitude are random, so they add as

$$\sum_t \Delta \delta u_l \sim \delta u_l \chi \sqrt{N}$$, where $N = t/\tau_A$ is the # of kicks

The cascade time τ_l is defined as the time to achieve an order unity change in the amplitude

So $\sum_t^{\tau_l} \Delta \delta u_l \sim \delta u_l \Rightarrow \tau_l \sim \tau_{eddy}^2 / \tau_A$

$$\therefore \; \epsilon \sim \frac{\delta u_l^2}{\tau_l} \sim \delta u_l^2 \frac{\delta u_l^2 l_\parallel}{v_A l_\perp^2} \sim \text{const}$$

$$\Rightarrow \delta u_l \sim (\epsilon v_A)^{1/4} l_\perp^{1/2} l_\parallel^{-1/4}$$

IK also assumed that $l_\parallel \sim l_\perp$ (isotropy) (VI)

So, (I) + (VI) $\Rightarrow \delta u_l \sim (\epsilon v_A)^{1/4} l^{1/4}$ or $\boxed{E \sim (\epsilon v_A)^{1/2} k^{-3/2}}$

IK spectrum

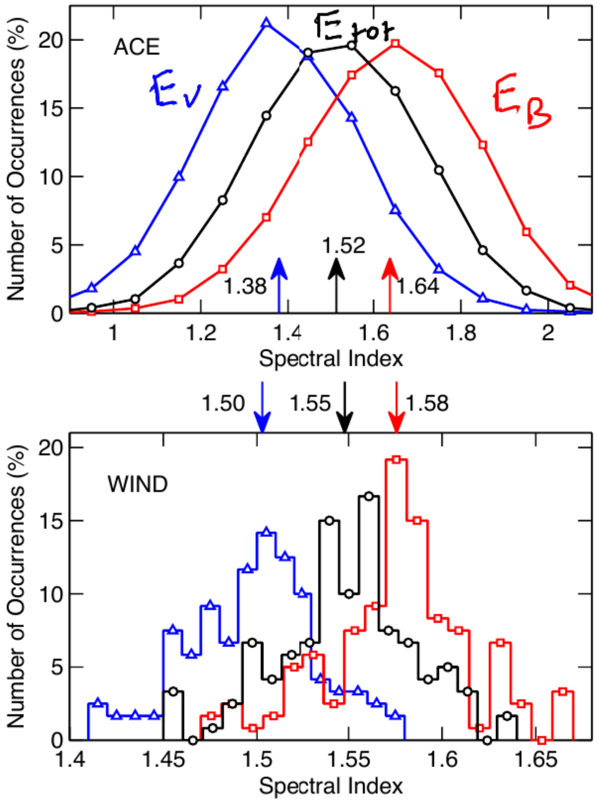
Also, recall that we are dealing with Alfvenic fluctuations, so $E_B \sim E_v$.

Is the weak interaction self-consistent, ie, does it hold at all scales in the inertial range?

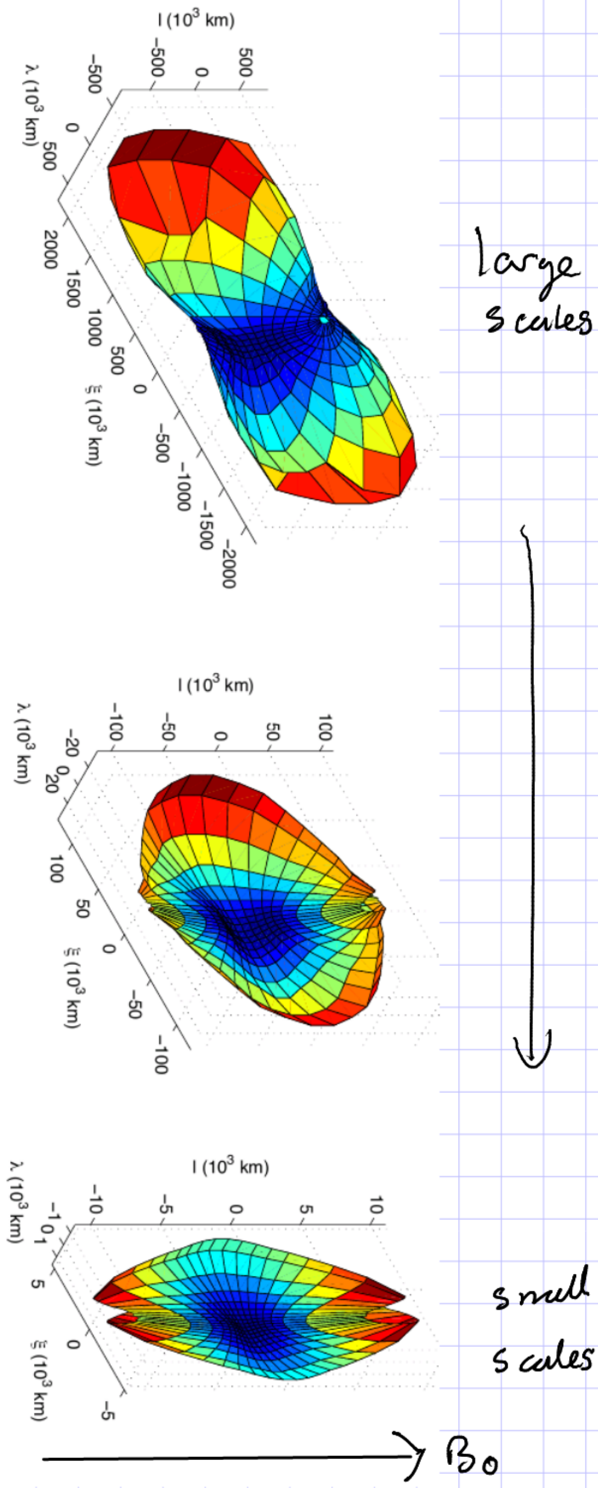
$$\chi \sim \frac{l_\parallel}{v_A} \frac{\delta u_l}{l_\perp} \sim \frac{\delta u_l}{v_A} \sim \frac{(\epsilon v_A)^{1/4}}{v_A} l^{1/4}$$

$$\epsilon \sim \frac{\delta u_L^2}{\tau_L} \sim \frac{\delta u_L^4}{v_A} \frac{1}{L} \Rightarrow \chi \sim \frac{\delta u_L}{v_A} \left(\frac{l}{L}\right)^{1/4}$$, which is small for $l \ll L$ provided $\delta u_L < v_A$, and it gets smaller with l

So, IK is self-consistent, and the community rejoiced... until high quality measurements become available. The observed spectra for the magnetic field are closer to $k^{-5/3}$ than $k^{-3/2}$ (1.666.. and 1.5 are hard to differentiate). Also, DNSs showed that $\ell_{\parallel} \sim \ell_{\perp}$ is not satisfied.



From Boldyrev et al (2011)
ACE = 10 yrs solar wind data
WIND = 11 yrs " " "



Surfaces of constant B measured using Ulysses
From Chen et al (2011)

Weak Turbulence (Montgomery & Turner 1981)

Let's see if we can fix IK based on data.

Consider the classic three wave interaction, 2 waves interact to produce a third.

We have two oppositely propagating Alfven waves with $\omega = |k_\parallel| v_A$. They must satisfy

Energy: $\omega(\vec{k}_1) + \omega(\vec{k}_2) = \omega(\vec{k}_3) \Rightarrow k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3}$

Momentum: $\vec{k}_1 + \vec{k}_2 = \vec{k}_3 \Rightarrow k_{\parallel 1} - k_{\parallel 2} = k_{\parallel 3}$

These can only be satisfied if $k_{\parallel 1}$ or $k_{\parallel 2} = 0$, take $k_{\parallel 2} = 0$

$\therefore$ $k_{\parallel 1} = k_{\parallel 3}$. There is no parallel cascade! And, weak turbulence is mediated by $k_\parallel = 0$ modes.

So, instead of (VI) $\ell_\perp \sim \ell_\parallel$, it should be (VI) $\ell_\parallel \sim L$

and $\delta u_\ell \sim (\epsilon v_A)^{1/4} \ell_\perp^{1/2} \ell_\parallel^{-1/4} \sim \left(\frac{\epsilon v_A}{L}\right)^{1/4} \ell_\perp^{1/2}$ (★) $\Rightarrow$ $\boxed{E \sim \left(\frac{\epsilon v_A}{L}\right)^{1/2} k_\perp^{-2}}$

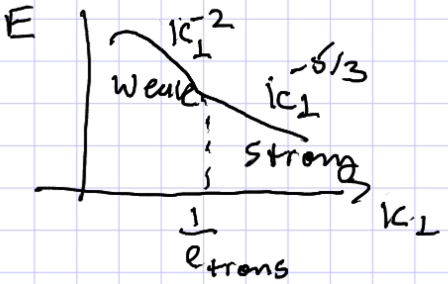
Weak turb. spectrum

Now, under this new assumption

$\chi \sim \frac{\delta u_\ell}{v_A}\left(\frac{L}{\ell_\perp}\right)^{1/2}$, which grows with decreasing $\ell_\perp$!

$\chi \sim 1$ when $\ell_\perp \sim \ell_{trans} \sim L \left(\frac{\delta u_L}{v_A}\right)^2$

So, at the transition scale, ℓ_{trans}, the weak turbulence assumption breaks down and it becomes strong, $\tau_{eddy} \sim \tau_A$



So, let's explore strong turbulence

Goldreich - Sridhar (1995)

Let's replace our anisotropy assumption by something less restrictive.

Critical balance: $\chi \sim 1$ (VI), i.e., $\tau_A \sim \tau_{eddy}$ or

$|\vec{z}^{\mp} \cdot \nabla \vec{z}^{\pm}| \sim |v_A \nabla_{\parallel} \vec{z}^{\pm}|$. So, $k_\perp \delta u_\ell \sim k_\parallel v_A$

Now, there is just one time scale $\tau_\ell \sim l_\perp / \delta u_\ell$

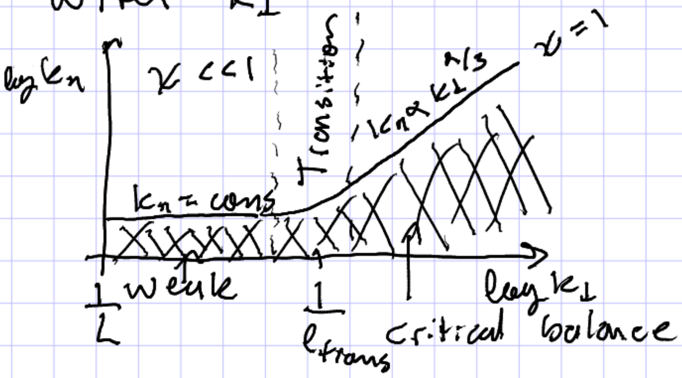
$$\therefore \epsilon \sim \frac{\delta u_\ell^2}{\tau_\ell} \sim \frac{\delta u_\ell^3}{\ell_\perp} \Rightarrow \delta u_\ell \sim (\epsilon \ell_\perp)^{1/3} \Rightarrow \boxed{E \sim \epsilon^{2/3} k_\perp^{-5/3}}$$

GS spectrum

Together with $\chi \sim 1$

$$k_\parallel v_A \sim k_\perp \delta u_\ell \sim \epsilon^{1/3} k_\perp^{2/3} \Rightarrow \boxed{k_\parallel \sim \frac{\epsilon^{1/3}}{v_A} k_\perp^{2/3}}$$ reduced parallel cascade

So, GS predict that the <u>wave vector</u> anisotropy grows with $k_\perp$



Since there is now a parallel cascade, we can also derive a parallel wavenumber spectrum

$$\epsilon \sim \frac{\delta u_\ell^2}{\tau_\ell} \sim \delta u_\ell^2 v_A k_\parallel \Rightarrow \delta u_\ell^2 \sim (\epsilon / v_A) k_\parallel^{-1/2}$$

$$\Rightarrow \boxed{E(k_\parallel) \sim \frac{\epsilon}{v_A} k_\parallel^{-2}}$$

<u>End of the inertial range</u> (assuming $l_\parallel$ or $l_\perp$ > kinetic scales)

We now have both viscosity and resistivity that could terminate our inertial range. So, we can construct two Reynold's type numbers

As in hydro $Re := \frac{\text{convection}}{\text{viscous}} = \frac{\delta u_L L}{\nu}$

Similarly, the magnetic Reynold's number is

$Re_m = \frac{\text{convection}}{\text{resistive}} = \frac{\delta u_L L}{\eta}$, this should not be confused with the Lundquist number $S := \frac{L V_A}{\eta}$, which relates the Alfven crossing time and resistive diffusion.

The magnetic Prandtl number is the ratio of

$Pr_m = \frac{Re_m}{Re} = \frac{\nu}{\eta}$ and characterizes the relative strength of viscous to magnetic diffusivity.

For simplicity, we'll assume $Pr_m \gg 1$.

So, at the viscous scale

$$\epsilon \sim \frac{\delta u_{\ell_\nu}^3}{\ell_{\perp\nu}} \sim \nu \frac{\delta u_{\ell_\nu}^2}{\ell_{\perp\nu}^2} \quad \text{and} \quad \delta u_{\ell_\nu}^2 \sim (\epsilon\, \ell_{\perp\nu})^{2/3}$$

$$\Rightarrow \ell_{\perp\nu} \sim \frac{\nu^{3/4}}{\epsilon^{1/4}} \quad \text{as before}$$

weak turbulence result see ⊛

$$\text{but} \quad \nu^{3/4} = Re^{-3/4}\, \delta u_L^{3/4} L^{3/4} \quad \text{and} \quad \epsilon^{1/4} \sim \frac{\delta u_L}{(V_A L)^{1/4}}$$

$$\Rightarrow \ell_{\perp\nu} \sim Re^{-3/4} L \underbrace{\left(\frac{V_A}{\delta u_L}\right)^{1/4}}_{\text{New}}$$

When is $\ell_{trans} \gg \ell_\perp \gg \ell_{\perp\nu}$ valid?

$$\ell_{trans} \sim L \left(\frac{\delta u_L}{V_A}\right)^2 \gg \ell_{\perp\nu} \sim Re^{-3/4} L \left(\frac{V_A}{\delta u_L}\right)^{1/4}$$

So, $Re \gg (V_A/\delta u_L)^3$ to have strong turbulence and $1 \gg \frac{\delta u_L}{V_A}$ must be true for weak turbulence

References

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