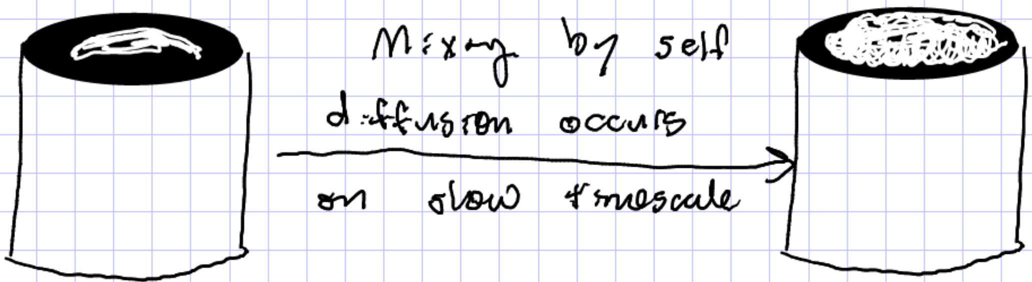
# I Introduction to turbulence and K41)

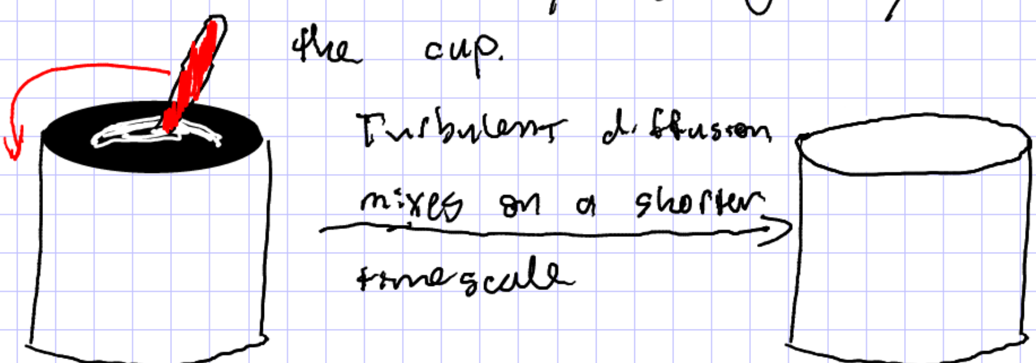
Turbulence is a chaotic flow regime characterized by diffusivity, rotationality, and dissipation.

## Cartoon picture of Turbulence

Mixing of cream (tracer) into coffee




Instead, we stir. Thereby injecting energy at the scale of the cup.




## Equations of Hydrodynamic Turbulence (Euler Eqns)

1) Incompressibility, $\rho = \text{const} \Rightarrow \nabla \cdot \vec{u} = 0$
2) Navier-Stokes (NS)

$$\frac{\partial \vec{u}}{\partial t} + \underbrace{\vec{u} \cdot \nabla \vec{u}}_{\substack{\text{Convection} \\ \text{(Nonlinear term)}}} = \underbrace{-\nabla\left(\frac{p}{\rho}\right)}_{\substack{\text{Thermal} \\ \text{forces}}} + \underbrace{\nu \nabla^2 \vec{u}}_{\text{Viscosity}} + \underbrace{\vec{f}}_{\substack{\text{External} \\ \text{forcing}}}$$

3) Energy: $\frac{\partial E}{\partial t} + \nabla \cdot [\vec{u}(E+p)] = 0$ ; $E = \frac{1}{2}\rho u^2 + \rho e$, $e$ = internal energy

Aside: when can we assume incompressibility?

1) $\Delta\rho/\rho \ll 1$ : Adiabatic change in $\Delta\rho \Rightarrow \Delta p = (\partial p/\partial\rho)_s \Delta\rho$

Bernoulli's eqn $\Rightarrow \Delta p \sim \rho u^2$ and $(\partial p/\partial\rho)_s \equiv c_s^2 = \gamma p/\rho$ Ideal gas

$\therefore \Delta\rho/\rho \ll 1 \Rightarrow u/c_s = Ma \ll 1$ $Ma :=$ Mach number

2) $\partial\rho/\partial t \ll \rho \nabla\cdot\vec{u}$ : $\frac{\partial\rho}{\partial t} \sim \frac{\Delta\rho}{t} \sim \frac{\rho u^2}{t c_s^2}$ ; $\rho\nabla\cdot\vec{u} \sim \rho u/l$

$\therefore \frac{\rho u^2}{t c_s^2} \ll \frac{\rho u}{l}$ Assuming $u \sim \frac{l}{t} \Rightarrow t \gg l/c_s \Rightarrow$ Information propagates instantaneously

Parameters of the system:

- characteristic (outer-scale) velocity, $u_0$ } external forcing fixes these*
- characteristic (outer-scale) length, $L$
- viscosity, $\nu$ set by molecular properties

* Outer-scale is also called the integral or autocorrelation scale

$$L^{int} = \int_0^\infty R_{ii}(r,t)\,dr \;;\; R_{ii} = \frac{\langle u_i(x_i,t)\, u_i(x_i+r,t)\rangle}{\langle u_i^2\rangle}$$

Let's compare 2 important terms in the NS eqn

$$\frac{\text{convection}}{\text{viscous}} \sim \frac{u_0^2/L}{\nu u_0/L^2} = \frac{u_0 L}{\nu} =: Re \quad \text{Reynold's number}$$

- When $Re$ is "small", viscous effects dominate and the flow is linear (laminar)
- When $Re$ is sufficiently large, the flow becomes chaotic $\rightarrow$ turbulent. How large is large enough depends on the system, but values of $10^2 - 10^4$ are typical.
- The transition between laminar and fully developed turbulence is very messy, so we will focus on <u>fully developed turbulence</u> only.

## Phenomenological picture of turbulence

- At each point in the fluid, the velocity is fluctuating around its mean value $\vec{u}_0$

$$\vec{u} = \vec{u}_0 + \delta \vec{u}$$

At the outer-scale, $\delta u_0 \sim \delta u_L$. Also, we can transform away the mean flow. So, we can redefine Re in terms of fluctuating quantities

$$Re = \frac{\delta u_L L}{\nu}$$

- Let us now consider what happens to the energy in this system $E = \frac{1}{2}\int d\vec{x}\, |\vec{u}|^2$

Dotting the NS eqn with $\vec{u}$ $\Rightarrow$

$$\frac{dE}{dt} = \underbrace{\nu \int d\vec{x}\, \vec{u}\cdot\nabla^2\vec{u}}_{\text{viscous dissipation}} + \underbrace{\int d\vec{x}\, \vec{u}\cdot\vec{f}}_{\text{rate of energy injection}}$$

If our system is in a stationary state (formally, we are considering the ensemble average $\equiv \langle\ \rangle$) then

$$\frac{d\langle E\rangle}{dt} = 0 = \nu \int d\vec{x} \langle \vec{u}\cdot\nabla^2\vec{u}\rangle + \underbrace{\int d\vec{x} \langle \vec{u}\cdot\vec{f}\rangle}_{=:\ \underset{\text{volume}}{V}\epsilon}$$

$$\therefore\ -\nu \int d\vec{x} \langle u\cdot\nabla^2 u\rangle = V\epsilon$$

In a steady state, the energy input rate, $\epsilon$, must match the dissipation rate.

- Let's now construct estimates for various quantities based on dimensional analysis

At the outer-scale, the turbulence is characterized by $u_0$, $L$. At smaller scales, we can consider the RMS velocity $u_\ell$ at scale $\ell$. Using just the velocity, length scale, and viscosity we now construct other important quantities

- Eddy turnover time: $t_\ell \sim \frac{\ell}{u_\ell}$

  at the outer-scale $t_L \sim \frac{L}{u_0}$

- Energy injection rate: $\epsilon \sim \frac{u_0^3}{L} \sim \frac{u_0^2}{t_L}$.

Using the energy injection rate, we can re-write

$t_L \sim \epsilon^{-1/3} L^{2/3}$ ①

- Dissipation time scale: $t_\ell^{diss} \sim \frac{\ell^2}{\nu}$ ②

- Viscous scale: Equating ① and ②

$$\Rightarrow \ell_\nu \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \sim L\, Re^{-3/4} \ll L$$

Note that this scale has multiple names: viscous scale, inner scale, dissipation scale, Kolmogorov scale are all common

Similarly, the outer-scale is often called the energy containing scale or integral scale

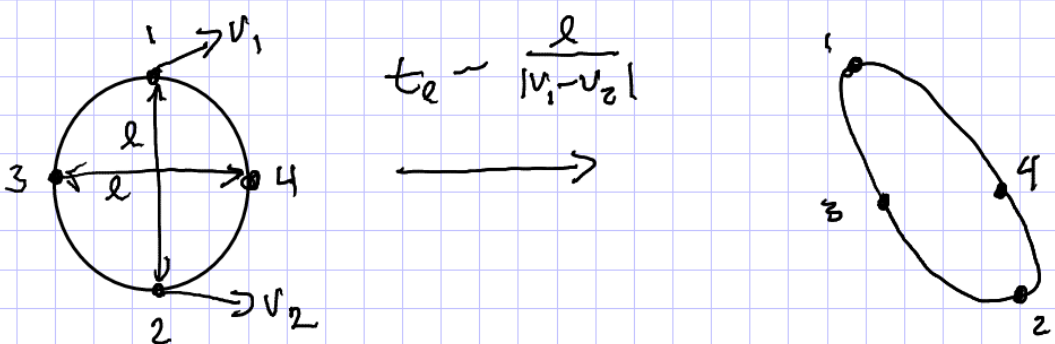
- Basic picture at this point

Energy injection → energy transport → energy dissipation

$L \gg \ell \gg \ell_\nu \sim L\, Re^{-3/4}$

Inertial Range

- Let's finish the cartoon picture of turbulence before we move on to K41.
- I defined the eddy turnover time above as $t_e \sim \frac{l}{u_l}$, but what does ~~it~~ mean? $t_e$ is the characteristic time for a structure of size $l$ to undergo a significant distortion due to the relative motion of its components




$\nabla \cdot u = 0$ ⇒ the area is conserved ⇒ if 1 & 2 separate, 3 & 4 become closer. ∴ $t_e$ is also the typical time for the transfer of energy from scale $l$ to a smaller scale. Also call this the cascade time.

## Kolmogorov 1941 Turbulence Theory

Before we discuss the theory, let's establish a baseline.

What observational facts do we know that we can use to constrain theory?

1) 2/3 law the mean square velocity increment, $\langle \delta u_l^2 \rangle$ between two points separated by $l$ scales as $\langle \delta u_l^2 \rangle \sim l^{2/3}$

2) Finite energy dissipation the energy dissipation is always positive and finite

- In 1922, Richardson conjectured that the energy transfer is local in space to the viscous scale


but this conjecture alone does not reproduce the observable aspects of turbulence above

- So, in 1941 Kolmogorov proposed the first theory that did explain 1) & 2) above. To do so, he assumed the following

0) Universality: The turbulence, inertial range, is independent of the particular forcing (and dissipation)
1) Locality of interactions
2) Homogeneity: No special points $\Rightarrow$ no intermittency
3) Isotropy: No special directions
4) Scale invariance: No special scales $\Rightarrow$ constant cascade rate, $\epsilon$.

- Scale invariance $\Rightarrow$ $\epsilon = \text{const}$, but we already argued that $\epsilon \sim \frac{u_0^3}{L}$. Since $\epsilon = \text{const}$,

$$\epsilon \sim \frac{\delta u_\ell^3}{\ell} \;\therefore\; \delta u_\ell \sim (\epsilon \ell)^{1/3} \Rightarrow \delta u_\ell^2 \sim \epsilon^{2/3} \ell^{2/3}$$

and we have the 2/3 law!

$\delta u_\ell \sim (\epsilon \ell)^{1/3}$ is referred to as the Kolmogorov–Obukhov law

- Energy spectra

In general, $\vec{k} = (k_x, k_y, k_z)$, so $E^{(3)}(\vec{k})$ is the 3D energy spectrum. The total energy is thus $E = \int d\vec{k}\, E^{(3)}(\vec{k})$. If the energy is isotropic in k space (k≤ 3), then we can use spherical coordinates

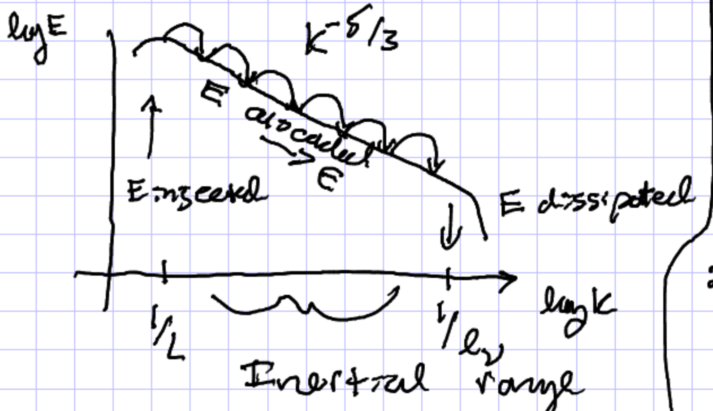
$$E = \iiint dk\, d\theta\, d\phi\, k^2 \sin\theta\, E^{(3)}(\vec{k}) = \int dk\, k^2 E^{(3)}(k) = \int dk\, E^{(1)}(k)$$

Where $E^{(1)}(k)$ is the 1D energy spectrum.

$$\therefore E = \delta u_\ell^2 \sim \epsilon^{2/3} \ell^{2/3} \sim \epsilon^{2/3} k^{-2/3} \sim \int_{k=1/\ell}^{\infty} dk\, k^2 E^{(3)} = \int dk\, E^{(1)}(k)$$

$$\Rightarrow \underbrace{E^{(1)} \sim \epsilon^{2/3} k^{-5/3}} \quad \text{and} \quad E^{(3)} \sim \epsilon^{2/3} k^{-11/3}$$

Normally use the 1D spectrum




What about scales $\ell < \ell_\nu$ and $\ell > L$?

1) $\ell > L$: $E = u_0^2 = \text{const} \Rightarrow$
$E^{(1)} \sim k^{-1}$

2) $\ell < \ell_\nu$: $\epsilon \sim \nu \vec{u} \cdot \nabla^2 \vec{u} \sim \nu \frac{\delta u_\ell^2}{\ell^2}$
$\Rightarrow \delta u_\ell \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} \ell \Rightarrow$
$E^{(1)} \sim k^{-3}$

<u>Inertial range</u> is the range of scales unaffected by driving or dissipation. The physics is assumed to be self-similar (fractal) here.

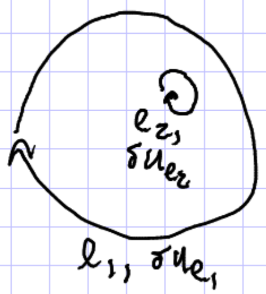
Note that $E \sim \ell^{2/3}$ dominated by large scales and gradients $\frac{\delta u_\ell}{\ell} \sim \ell^{-2/3}$ dominated by small scales $\Rightarrow$ viscous cutoff.

Also, the cascade time, $t_\ell \sim \frac{\ell}{\delta u_\ell} \sim \ell^{2/3}$ decreases with scale.

## "Verifying" that the cascade is local

Consider motions at scales $l_1$ and $l_2$.

1) Can large scale motion shear apart small scales before they cascade?



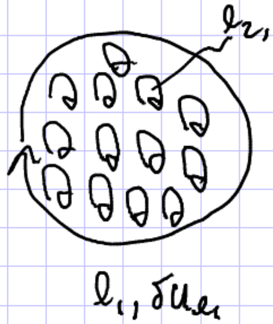

Shearing time $= t_S \sim \dfrac{l_1}{\delta u_{l_1}}$

cascade time for $l_2$ $\quad t_{l_2} \sim \dfrac{l_2}{\delta u_{l_2}}$

$$\frac{t_S}{t_{l_2}} \sim \frac{l_1 \, \delta u_{l_2}}{l_2 \, \delta u_{l_1}} \sim \frac{l_1 \left[\delta u_{l_1} \left(\frac{l_2}{l_1}\right)^{1/3}\right]}{l_2 \, \delta u_{l_1}}$$

$\sim \left(\dfrac{l_1}{l_2}\right)^{2/3} \gg 1 \Rightarrow$ shearing by large scales not important.

2) Can small scale eddies diffuse the large eddies before they cascade?



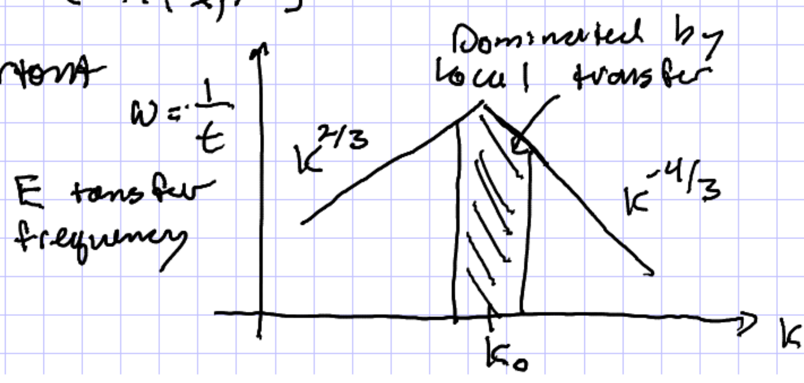

Diffusion coefficient due to eddies of size $l_2$: $D = \dfrac{l_2^2}{t_{l_2}} = l_2 \, \delta u_{l_2}$

Time to diffuse distance $l_1$ is

$t_D \sim \dfrac{l_1^2}{D} \sim \dfrac{l_1^2}{l_2 \, \delta u_{l_2}}$

$$\frac{t_D}{t_{l_1}} \sim \frac{l_1^2 \, \delta u_{l_1}}{\cancel{l_1} \, l_2 \, \delta u_{l_2}} \sim \frac{l_1}{l_2} \frac{\delta u_{l_1}}{\left[\delta u_{l_1} \left(\frac{l_2}{l_1}\right)^{1/3}\right]} = \left(\frac{l_1}{l_2}\right)^{4/3} \gg 1 \Rightarrow$$

diffusion is unimportant



## Additional Reading

1) Uriel Frisch "Turbulence: the Legacy of A.N. Kolmogorov" 1996

2) Landau & Lifshitz "Fluid Mechanics" 1987
chapter 3

3) Kolmogorov 1941. Translation by Levin 1951
"The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers"

4) P.A. Davidson "Turbulence: An Introduction" 2004