Sweet-Parker reconnection: scaling for reconnection rates

Set-up: Let's assume a 2D geometry with the direction of the inflowing fields in the $\pm x$ direction and the direction of the flow is $\pm y$. The width of the reconnection layer is $2L$ and thickness is 2δ. This is the famous picture from Petschek. The key to the Sweet-Parker model is to consider the inner and outer domains separately since different physics applies (standard trick called boundary layer theory).

Continuity: From the continuity equation (conservation of mass, incompressibility) it's easy to show that:

$$u_{in} L = u_{out}\delta \rightarrow \frac{u_{in}}{u_{out}} = \frac{\delta}{L}.$$

Ohm's law: Outside the layer, the field lines are straight ($\nabla \times B = 0$) so there's no current and we have for the z-component of Ohm's law:

$$E_z + u_{in} B_x = 0.$$

Inside the layer, there's different physics. The velocity has stagnated to zero and there's a big $\nabla \times B$ so there's a big current. We have for the z-component of Ohm's law:

$$E_z = \eta J_z.$$

We can connect the two relationships if we assume the flow is steady state. In other words, there's a continuous flow in and out of the layer but if you close your eyes and open them a little later, the picture looks the same. From Faraday's law we have

$$\nabla \times E = -\frac{\partial B}{\partial t} = 0$$

in other words, the electric field E_z inside and outside the layer is the same (as long as the flow in and out of the layer is steady)!

Conservation of energy: From energy balance (assuming all the B_x field is annihilated) we find:

$$\frac{B_x^2}{2\mu_0} = \frac{\rho u_{out}^2}{2} \rightarrow u_{out} = u_{Alfven} = \frac{B_x}{\sqrt{\mu_0 \rho}}$$

Ampere's law: Now let's integrate around the layer $\int B \cdot dl = \mu_0 I$ and we find:

$$B_x(4L) = \mu_0 J_z (2L)(2\delta) \rightarrow B_x = \mu_0 J_z \delta.$$

If we combine the results from Ampere's law and Ohm's law, we find

$$u_{in} = \frac{E_z}{B_x} = \frac{\eta J_z}{\mu_0 J_z \delta} = \frac{\eta}{\mu_0 \delta}.$$

Another way to write this is

$$\frac{u_{in}\mu_0\delta}{\eta} \equiv R_m = 1!$$

The interpretation is that the inflow speed and the layer width adjust themselves such that the magnetic Reynold's number (ratio of convection to diffusion) is unity (based on the layer width). In other words, things adjust themselves until magnetic flux is annihilated as fast as plasma can be exhausted out the sides. Notice that if δ is very small, then the flow gets backed up on the inflow side. A broad or fanned outflow region is analogous to having a big exhaust manifold on a high performance car engine... the faster the exhaust can get out of the way, the more energy the engine can convert.

Reconnection rate: My trick for extracting the reconnection rate is to write down the square of the inflow speed using the two relations (from continuity and Ampere):

$$u_{in}^2 = (u_{out}\frac{\delta}{L})(\frac{\eta}{\mu_0\delta}) = u_{out}^2(\frac{\eta}{u_{Alfven}\mu_0 L})$$

(here I've used u_{out} and u_{Alfven} interchangeably). The reconnection rate is often couched in terms of the normalized inflow speed because recall that $u_{in} \times B_x$ is the electric field which is related to the consumption of flux at the x-point (from Faraday's law). In any case, the reconnection rate is written:

$$\frac{u_{in}}{u_{out}} = \frac{1}{\sqrt{S}}$$

where S is called the Lundquist number (or the magnetic Reynold's number based on the Alfvén speed). Note that S is defined even if there's no flow. Furthermore, S is typically the largest Reynold's number in the problem since u_{Alfven} is the largest velocity we expect in MHD (from energy conservation). More sophisticated models (Petschek or Vasyliunas) predict a faster reconnection rate than $1/\sqrt{S}$ (more like 1/log(S)). There appears to be a bound on the reconnection rate:

$$\frac{1}{log(S)} \leq \frac{u_{in}}{u_{out}} \leq \frac{1}{\sqrt{S}}.$$

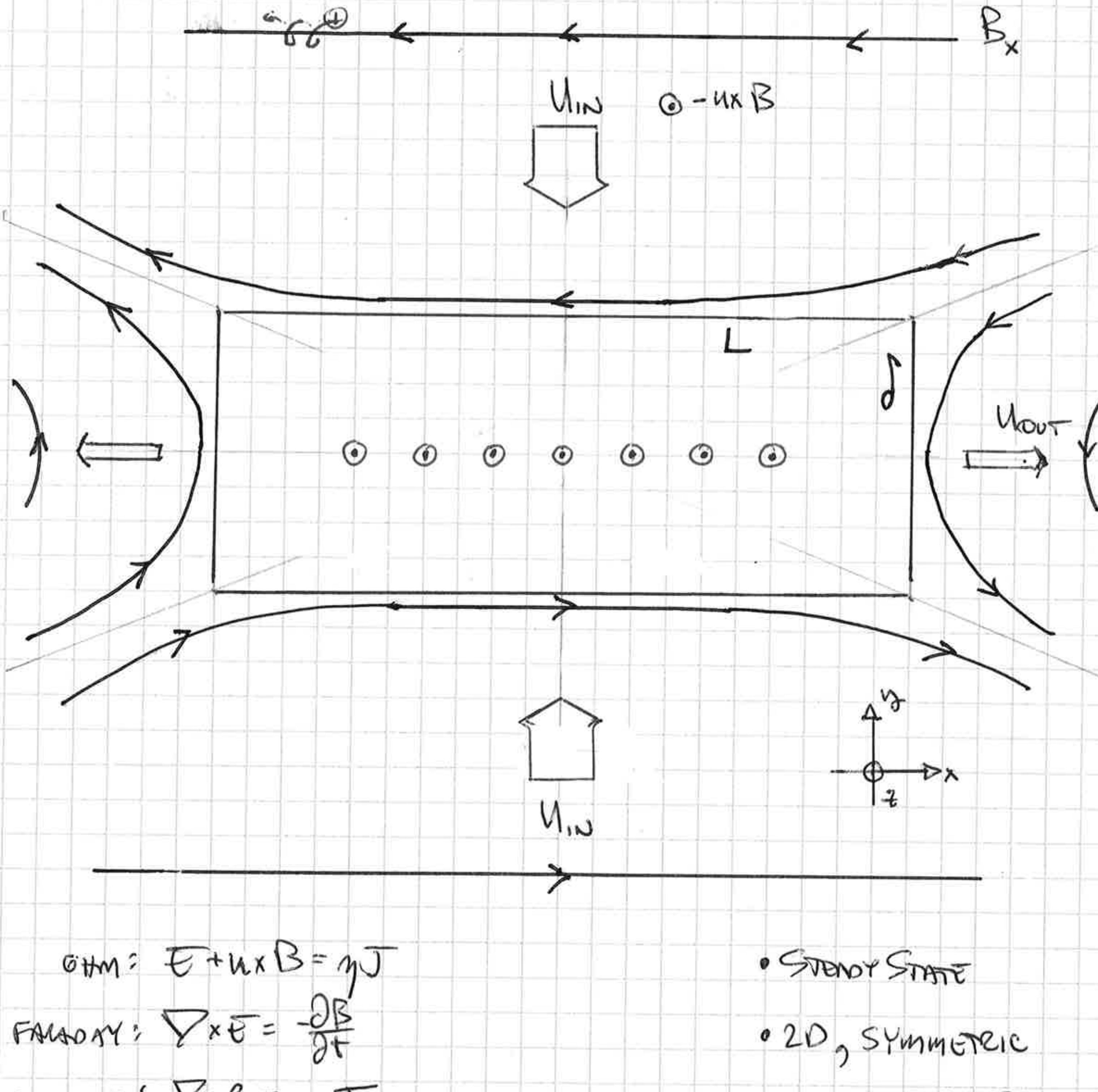
Simulations (as well as our experiments and some observations) show a "ubiquitous" reconnection rate of about 0.1 over a wide range of parameters (ie $10^3 \leq S \leq 10^9$).

Simple diffusion: Finally, if you consider the case with no flow (that is, two semi-infinite slabs of magnetofluid with oppositely directed magnetic field that are just lying together), then the Ohm's law is particularly simple: $E = \eta J$. Ampere's law tells you that $\nabla \times B$ generates a thin current sheet between the slabs (a δ-function actually). If you take the curl of Ohm's law (and invoke Faraday and Ampere laws) you get a diffusion equation (actually the diffusive form of the induction equation):

$$\frac{\partial B}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 B.$$

The initial condition is a step function in B and (therefore) a delta function in J. As time progresses, the step function smooths out and the delta function broadens. The scaling for this process is 1/S (ie very slow).

Reconnection Geometry (Sweet-Parker)



Ohm: $E + u\times B = \eta J$

Faraday: $\nabla\times E = -\frac{\partial B}{\partial t}$

Ampère: $\nabla\times B = \mu_0 J$

Incompressible: $\nabla\cdot u = 0$

- Steady State
- 2D, Symmetric
- Incompressible
- MHD (no ion, e^-)
- No Heating, No Viscosity